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# Bubble towers for supercritical semilinear elliptic equations

Yuxin Ge[a], Ruihua Jing[a,b], Frank Pacard[a,*]

[a]Département de Mathématiques, Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, Université Paris XII-Val de Marne, 61 avenue du Général de Gaulle, 94010 Créteil, Cedex, France

[b]Department of Mathematics, East China Normal University, 200062 Shanghai, People's Republic of China

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**Abstract**

We construct positive solutions of the semilinear elliptic problem $\Delta u + \lambda u + u^p = 0$ with Dirichet boundary conditions, in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ $(N \geqslant 4)$, when the exponent $p$ is supercritical and close enough to $\frac{N+2}{N-2}$ and the parameter $\lambda \in \mathbb{R}$ is small enough. As $p \to \frac{N+2}{N-2}$, the solutions have multiple blow up at finitely many points which are the critical points of a function whose definition involves Green's function. Our result extends the result of Del Pino et al. (J. Differential Equations 193(2) (2003) 280) when $\Omega$ is a ball and the solutions are radially symmetric.


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* Corresponding author.
*E-mail addresses:* ge@univ-paris12.fr (Y. Ge), jing@univ-paris12.fr (R. Jing), pacard@univ-paris12.fr (F. Pacard).

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## 1. Introduction

In this paper, we consider the semilinear elliptic problem

$$\begin{cases} \Delta u + \lambda u + u^p = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega$ is a bounded regular domain in $\mathbb{R}^N$, $N \geqslant 4$, the parameter $\lambda \in \mathbb{R}$ and the exponent $p$ is larger than

$$p_N := \frac{N+2}{N-2},$$

the critical Sobolev exponent.

When $p = p_N$, Brezis and Nirenberg [3] have proved that (1) admits a solution provided $0 < \lambda$ is less than the first eigenvalue of the Laplacian on $\Omega$ with 0 Dirichlet boundary condition. Direct application of Pohozaev's identity [12] shows that solutions of (1) do not exist when $\lambda \leqslant 0$, $p \geqslant p_N$ and $\Omega$ is a star-shaped domain.

In this paper, we are interested in the existence of solutions of (1) in the case where $p$ is larger than the critical Sobolev exponent. When $\Omega$ is the unit ball it is easy to check that there exist radially symmetric positive solutions of

$$\Delta u + u^p = 0,$$

which have multiple blow up at the origin as the exponent $p$ tends to $p_N$ (we do not assume Dirichlet boundary condition here). We discuss this result in Section 4. In a recent paper [5], Del Pino et al. have proved that a similar result was also true for (1). These solutions which have multiple blow up at some points in $\Omega$ will be referred to as "bubble tree solutions". We are interested in the existence of these bubble tree solutions when $\Omega$ is arbitrary.

## 2. Statement of the result

Let $G$ denote Green's function for the Laplace operator with Dirichlet boundary condition on $\Omega$ and let $H$ denote Robin's function, i.e. the regular part of Green's function. Namely,

$$G(y,z) := |y-z|^{2-N} - H(y,z)$$

for $(y,z) \in \Omega \times \Omega$. Observe that $\Delta_y H = 0$ in $\Omega \times \Omega$ and $G = 0$ on $\partial(\Omega \times \Omega)$.