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Journal of Functional Analysis 221 (2005) 303–322

JOURNAL OF Functional Analysis

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Geometric quantization, complex structures and the coherent state transform

Carlos Florentino[a], Pedro Matias[b], José Mourão[a,*], João P. Nunes[a]

[a]*Department of Mathematics, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*
[b]*Center for Mathematics and its Applications, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

Received 20 February 2004; received in revised form 12 July 2004; accepted 21 October 2004
Communicated by L. Gross
Available online 23 January 2005

Abstract

It is shown that the heat operator in the Hall coherent state transform for a compact Lie group K (J. Funct. Anal. 122 (1994) 103–151) is related with a Hermitian connection associated to a natural one-parameter family of complex structures on T^*K. The unitary parallel transport of this connection establishes the equivalence of (geometric) quantizations of T^*K for different choices of complex structures within the given family. In particular, these results establish a link between coherent state transforms for Lie groups and results of Hitchin (Comm. Math. Phys. 131 (1990) 347–380) and Axelrod et al. (J. Differential Geom. 33 (1991) 787–902).

Keywords: Geometric quantization; Coherent state transform for Lie groups

1. Introduction

In the present paper, we relate the appearance of the heat operator in the Hall coherent state transform (CST) for a compact connected Lie group K [Ha1] with a one-parameter family of complex structures on the cotangent bundle T^*K, in the framework of geometric quantization. The heat equation appears also in the quantization

* Corresponding author.
E-mail addresses: cfloren@math.ist.utl.pt (C. Florentino), pmatias@math.ist.utl.pt (P. Matias), jmourao@math.ist.utl.pt (J. Mourão), jpnunes@math.ist.utl.pt (J.P. Nunes).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.10.021

of $\mathbb{R}^{2n}$ and of Chern–Simons theories [AdPW,Hi] and in the related theory of theta functions, where it is associated with the so-called Knizhnik–Zamolodchikov–Bernard–Hitchin (KZBH) connection [Fa,Las,Ra,FMN]. The general case was studied from a cohomological point of view in [Hi].

Our main motivation is to give a differential geometric interpretation to the appearance of the heat equation in the Kähler quantization of T^*K, thus answering a question raised in [Ha3, §1.3]. This interpretation is based in the projection of the prequantization connection to the quantum sub-bundle in geometric quantization, as proposed in [AdPW]. The main advantage of this approach consists in the fact that it ensures that the quantum connection is Hermitian. Our method is also complementary to the one of Thiemann [Th1,Th2] where he considers generalized canonical transformations generated by complex-valued functions on the phase space. The heat equation appears then naturally as the Schrödinger equation for these complex Hamiltonians or complexifiers.

As shown in [AdPW, §1a], for $\mathbb{R}^{2n}$ the heat equation is associated with independence of the quantization with respect to the choice of a complex structure within the family of complex structures which are invariant under translations.

We consider on T^*K a one-parameter family of complex structures $\{J_s\}_{s\in\mathbb{R}_+}$ induced by the diffeomorphisms

$$\begin{array}{ccccccc} T^*K & \simeq & K\times\mathfrak{K}^* & \simeq & K\times\mathfrak{K} & \stackrel{\psi_s}{\to} & K_{\mathbb{C}} \\ & & & & (x,Y) & \mapsto & xe^{isY}, \end{array} \tag{1.1}$$

where $K_{\mathbb{C}}$ is the complexification of K. Here, we identify T^*K with $K\times\mathfrak{K}^*$ by means of left-translation and then with $K\times\mathfrak{K}$ by means of an Ad-invariant inner product $(\cdot\,,\cdot)$ on $\mathfrak{K}=\mathrm{Lie}(K)$.

Together with the canonical symplectic structure ω, the pair (ω, J_s) defines on T^*K a Kähler structure for every $s\in\mathbb{R}_+$. Hall has shown [Ha3] that, when one considers geometric quantization of T^*K, the CST, which has been proved to be unitary in [Ha1], gives (up to a constant factor) the pairing map between the vertically polarized Hilbert space and the Kähler polarized Hilbert space, provided that one takes into account the half-form correction.

The family of complex structures $\{J_s\}$ is generated by the flow of the vector field $v=\sum_j y^j\frac{\partial}{\partial y^j}$. This vector field is not Hamiltonian but it is given by $v=J_s(\sum_j sy^jX_j)$, where we note that $i\sum_j y^jX_j$ is the Hamiltonian vector field for the complex Hamiltonian $\frac{i}{2}|Y|^2$. This is the complex Hamiltonian used by Thiemann [Th1,Th2] (see also [Ha3]) to generate quantum states in the holomorphic polarization from the vertically polarized ones. The actions of the vector field v and of the Hamiltonian vector field corresponding to $s\frac{i}{2}|Y|^2$ coincide on J_s-holomorphic functions. This explains the relation between our formalism and the formalism of complexifiers proposed by Thiemann.

In order to associate the heat operator with an Hermitian connection, we collect the prequantum and quantum Hilbert spaces for all $s\in\mathbb{R}_+$ in a prequantum and a quantum Hilbert bundles over $\mathbb{R}_+$,

$$\mathcal{H}^{\mathrm{prQ}}\to\mathbb{R}_+$$