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Saturating constructions for normed spaces II

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Abstract

We prove several results of the following type: given finite-dimensional normed space V possessing certain geometric property there exists another space X having the same property and such that (1) $\log \dim X = O(\log \dim V)$ and (2) every subspace of X, whose dimension is not "too small", contains a further well-complemented subspace nearly isometric to V. This sheds new light on the structure of large subspaces or quotients of normed spaces (resp., large sections or linear images of convex bodies) and provides definitive solutions to several problems stated in the 1980s by Milman.

Keywords: Normed spaces; Subspaces; Quotients; Saturation; Cotype; K-convexity; Global properties; Random convex bodies

1. Introduction

This paper continues the study of the *saturation phenomenon* that was discovered in [ST] and of the effect it has on our understanding of the structure of high-dimensional

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normed spaces and convex bodies. In particular, we obtain here a dichotomy-type result which offers a near definitive treatment of some aspects of the phenomenon. We sketch first some background ideas and hint on the broader motivation explaining the interest in the subject.

Much of geometric functional analysis revolves around the study of the family of subspaces (or, dually, of quotients) of a given Banach space. In the finite-dimensional case this has a clear geometric interpretation: a normed space is determined by its unit ball, a centrally symmetric convex body, subspaces correspond to sections of that body, and quotients to projections (or, more generally, linear images). Such considerations are very natural from the geometric or linear-algebraic point of view, but they also have a bearing on much more applied matters. For example, a convex set may represent all possible states of a physical system, and its sections or images may be related to approximation or encoding schemes, or to results of an experiment performed on the system. It is thus vital to know to what degree the structure of the entire space (resp., the entire set) can be recovered from the knowledge of its subspaces or quotients (resp., sections/images). At the same time, one wants to detect some possible regularities in the structure of subspaces which might have not existed in the whole space.

A seminal result in this direction is the 1961 Dvoretzky theorem, with the 1971 strengthening due to Milman, which says that every symmetric convex body of large dimension n admits central sections which are approximately ellipsoidal and whose dimension k is of order $\log n$ (the order that is, in general, optimal). Another major result was the discovery of Milman [M2] from the mid 1980s that *every* n-dimensional normed space admits a *subspace of a quotient* which is “nearly” Euclidean and whose dimension is $\geqslant \theta n$, where $\theta \in (0,1)$ is arbitrary (with the exact meaning of “nearly” depending only on θ). Moreover, a byproduct of the approach from [M2] was the fact that every n-dimensional normed space admits a “proportional dimensional” quotient of bounded *volume ratio*, a volumetric characteristic of a body closely related to cotype properties (we refer to [MS1,T,P2] for definitions of these and other basic notions and results that are relevant here). This showed that one can get a very essential regularity in a global invariant of a space by passing to a quotient *or* a subspace of dimension, say, approximately $n/2$. It was thus natural to ask whether similar statements may be true for other related characteristics. This line of thinking was exemplified in a series of problems posed by Milman in his 1986 ICM Berkeley lecture [M3].

The paper [ST] elucidated this circle of ideas and, in particular, answered some of the problems from [M3]. A special but archetypal case of the main theorem from [ST] showed the existence of an n-dimensional space Y whose *every* subspace (resp., *every* quotient) of dimension $\geqslant n/2$ contains a further 1-complemented subspace isometric to a preassigned (but a priori arbitrary) k-dimensional space V, as long as k is at most of order $\sqrt{n}$. In a sense, Y was *saturated* with copies of the V. This led to the discovery of the following phenomenon: passing to large subspaces or quotients *cannot*, in general, erase k-dimensional features of a space if k is below certain threshold value depending on the dimension of the initial space and the exact meaning of “large”. In the particular case stated above, i.e., that of “proportional” subspaces or quotients, the threshold dimension was (at least) of order $\sqrt{n}$, and “impossibility to erase” meant that every such subspace (resp., quotient map) preserved a copy of the given V.