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# Dixmier traces as singular symmetric functionals and applications to measurable operators

Steven Lord[a,1], Aleksandr Sedaev[b,2], Fyodor Sukochev[a,*]

[a]*School of Informatics and Engineering, Flinders University of South Australia, Bedford Park 5042, Australia*
[b]*Department of Mathematics, Voronezh State University of Architecture and Construction, 20-letiya Oktyabrya 84, Voronezh 394006, Russia*

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**Abstract**

We unify various constructions and contribute to the theory of singular symmetric functionals on Marcinkiewicz function/operator spaces. This affords a new approach to the non-normal Dixmier and Connes–Dixmier traces (introduced by Dixmier and adapted to non-commutative geometry by Connes) living on a general Marcinkiewicz space associated with an arbitrary semifinite von Neumann algebra. The corollaries to our approach, stated in terms of the operator ideal $\mathscr{L}^{(1,\infty)}$ (which is a special example of an operator Marcinkiewicz space), are: (i) a new characterization of the set of all positive measurable operators from $\mathscr{L}^{(1,\infty)}$, i.e. those on which an arbitrary Connes–Dixmier trace yields the same value. In the special case, when the operator ideal $\mathscr{L}^{(1,\infty)}$ is considered on a type $I$ infinite factor, a bounded operator $x$ belongs to $\mathscr{L}^{(1,\infty)}$ if and only if the sequence of singular numbers $\{s_n(x)\}_{n\geqslant 1}$ (in the descending order and counting the multiplicities) satisfies $\|x\|_{(1,\infty)} := \sup_{N\geqslant 1} \frac{1}{\operatorname{Log}(1+N)} \sum_{n=1}^{N} s_n(x) < \infty$. In this case, our characterization amounts to saying that a positive element $x \in \mathscr{L}^{(1,\infty)}$ is measurable if and only if $\lim_{N\to\infty} \frac{1}{\operatorname{Log} N} \sum_{n=1}^{N} s_n(x)$ exists; (ii) the set of Dixmier traces and the

*Corresponding author. Fax: +61 882012904.
*E-mail addresses:* sed@vmail.ru (A. Sedaev), sukochev@infoeng.flinders.edu.au (F.A. Sukochev).
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set of Connes–Dixmier traces are norming sets (up to equivalence) for the space $\mathscr{L}^{(1,\infty)}/\mathscr{L}_0^{(1\infty)}$, where the space $\mathscr{L}_0^{(1,\infty)}$ is the closure of all finite rank operators in $\mathscr{L}^{(1,\infty)}$ in the norm $\|\cdot\|_{(1,\infty)}$.


*Keywords:* Non-normal (Dixmier) traces; Singular symmetric functionals; Banach limits; Marcinkiewicz spaces; Non-commutative geometry

## 0. Introduction

In [3] Dixmier proved the existence of non-normal traces on the von Neumann algebra $B(H)$. Dixmier's original construction involves singular dilation invariant positive linear functionals $\omega$ on $\ell^\infty(\mathbb{N})$. This construction was altered by Connes [2] (see also Definition 5.2 below) who defined non-normal traces via the composition of the Cesaro mean and a state on $C_b([0,\infty))/C_0([0,\infty))$. In [4–6] the traces of Dixmier [3] were broadly generalized as singular symmetric functionals on Marcinkiewicz function (respectively, operator) spaces $M(\psi)$ on $[0,\infty)$ (respectively, on a semifinite von Neumann algebra). The symmetric functionals in [5,6] involve Banach limits, that is, singular translation invariant positive linear functionals $L'$ on $\ell^\infty(\mathbb{N})$. We extend the construction of Dixmier in Definition 1.7 and Connes in Definition 5.2 (verified in Theorem 6.3) by extending the notion of Banach limits to $C_b([0,\infty))$.

The identification of the commutative specialization of (Connes-)Dixmier traces as singular symmetric functionals has some pivotal consequences. The established theory of Banach limits [10] and singular symmetric functionals on Marcinkiewicz spaces [4–6] can be applied to questions concerning the (Connes-) Dixmier trace, a central notion in Connes' non-commutative geometry [2]. Conversely, ideas in Connes' non-commutative geometry, such as measurability of operators [2, IV.2.$\beta$, Definition 7], lend themselves to generalization to abstract Marcinkiewicz spaces (Definitions 3.2 and 3.5). As a result, we have been able to present a new characterization of measurable operators (see Theorem 5.12, Remark 5.13 and Theorem 6.6).

The paper is structured as follows:

Section 1 introduces Banach limits, almost convergence (extending the notions of Lorentz [10]) and the theory of singular symmetric functionals on the Marcinkiewicz space $M(\psi)$ defined by a concave function $\psi$ [4,5]. The construction of singular symmetric functionals on $M(\psi)$ [5] (Definition 1.6 below) is extended by Definition 1.7.

Section 2 introduces sufficient conditions to identify the singular symmetric functionals of [5] with those of Definition 1.7, see Theorems 2.3 and 2.7. A result in [5], on the Riesz semi-norm of a function $x$ in a Marcinkiewicz space $M(\psi)$ as the supremum of the values $\{f(x)\}$ where $\{f\}$ is a set of singular symmetric functionals on $M(\psi)$, is extended in Theorem 2.8.

Section 3 contains an analysis of various notions of a measurable element of a Marcinkiewicz space $M(\psi)$, introduced in Definitions 3.2 and 3.5, and their coincidence (Theorem 3.7 and Corollary 3.9, see also Theorem 3.14).

The results of Sections 2 and 3 concern singular symmetric functionals on $M(\psi)$ parameterized by the set of strictly increasing, invertible, differentiable and unbounded