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Schilder theorem for the Brownian motion on the diffeomorphism group of the circle

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Abstract

We prove a large deviation principle for flows associated to stochastic differential equations with non-Lipschitz coefficients. As an application we establish a Schilder Theorem for the Brownian motion on the group of diffeomorphisms of the circle.

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1. Introduction

This paper is a continuation of our previous work [17]. Our motivation is to prove a Schilder theorem on the asymptotic behavior for the Brownian motion on the group of homeomorphisms of the circle which is constructed by Malliavin [16].

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Recall that the original Schilder theorem [18] states that if B is the real Brownian motion and $C_0[0,1]$ the space of real continuous functions defined on [0,1], null at 0, and endowed with the uniform convergence norm, then for any Borel set $A \subset C_0[0,1]$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P\{\varepsilon B \in A\} \leqslant -\Lambda(\bar{A}),$$

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log P\{\varepsilon B \in A\} \geqslant -\Lambda(A^0),$$

where $\bar{A}$ and A^0 denote the closure and interior of A respectively, and $\Lambda(A) := \inf_{\gamma\in A} \lambda(\gamma)$ with

$$\lambda(\gamma) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{\gamma}(s)|^2\, ds, & \gamma \text{ absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases} \tag{1}$$

This result was then generalized by Freidlin and Wentzell in their famous paper [10] by considering the Itô equation

$$\begin{cases} dX^\varepsilon(t) = \varepsilon\sigma(X^\varepsilon(t))\, dw_t + b(X^\varepsilon(t))\, dt \quad 0 \leqslant t \leqslant 1, \\ X^\varepsilon(0) = x. \end{cases} \tag{2}$$

They proved, under broad conditions (mainly the Lipschitzness and the boundedness of the coefficients), that for every Borel set A of $C_x([0,1],\mathbb{R}^d)$,

$$-\Lambda(A^0) \leqslant \liminf_{\varepsilon\to 0} \varepsilon^2 \log P(X_x^\varepsilon \in A) \leqslant \limsup_{\varepsilon\to 0} \varepsilon^2 \log P(X_x^\varepsilon \in A) \leqslant \Lambda(\bar{A}),$$

where $C_x([0,1],\mathbb{R}^d)$ is the space of continuous functions defined on $[0,1]$ and valued in $\mathbb{R}^d$, null at 0. Here also, the closure and the interior are taken in the topology of uniform convergence.

Changing the angle we can let x run over $\mathbb{R}^d$ and regard the solution as a random variable taking values in $C([0,1], D(\mathbb{R}^d))$ where $D(\mathbb{R}^d)$ is the space of C^∞ functions on $\mathbb{R}^d$ endowed with the topology of compact uniform convergence of derivatives of all orders, provided that the coefficients are smooth and of linear growth (see [12,14,15]). Thus one can also consider the large deviation principle (LDP in abbreviation) in this context, namely the asymptotic estimates of probabilities $P(X^\varepsilon \in A)$ where $A \in \mathcal{B}(C([0,1], D(\mathbb{R}^d)))$. See e.g. [2,11] among others for studies along this line.

Now let us return to the Brownian motion on the group of homeomorphisms of the circle. It is defined by a SDE as in (20) below. The main features of this equation are

(i) The coefficients are not Lipschitz, not to mention differentiable;
(ii) Infinitely many driving Brownian motions are involved.

To deal with such an equation we will first work in a more general context by considering flows generated by SDE in $\mathbb{R}^d$ with non-Lipschitz coefficients and driven