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Fractal entropies and dimensions for microstates spaces☆

Kenley Jung

Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720-3840, USA

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Abstract

Using Voiculescu's notion of a matricial microstate we introduce fractal dimensions and entropies for finite sets of selfadjoint operators in a tracial von Neumann algebra. We show that they possess properties similar to their classical predecessors. We relate the new quantities to free entropy and free entropy dimension and show that a modified version of free Hausdorff dimension is an algebraic invariant. We compute the free Hausdorff dimension in the cases where the set generates a finite-dimensional algebra or where the set consists of a single selfadjoint. We show that the Hausdorff dimension becomes additive for such sets in the presence of freeness.

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1. Introduction

Voiculescu's microstate theory has settled some open questions in operator algebras. With it he shows in [10] that a free group factor does not have a regular diffuse hyperfinite subalgebra (the first known kind with separable predual). Using similar techniques Ge shows in [2] that a free group factor cannot be decomposed into a

E-mail address: factor@math.berkeley.edu (K. Jung).
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tensor product of two infinite dimensional von Neumann algebras (again the first known example with separable predual). Both results rely upon the microstate theory and the (modified) free entropy dimension. Free entropy dimension is a number associated to an n-tuple of selfadjoint operators in a tracial von Neumann algebra. It is an analogue of Minkowski dimension and as such it can be reformulated in terms of metric space packings.

Unfortunately it is not known whether δ_0 is an invariant of von Neumann algebras, that is, if $\{b_1, \ldots, b_p\}$ is a set of selfadjoint elements in M which generates the same von Neumann algebra as $\{a_1, \ldots, a_n\}$, then is it true that

$$\delta_0(a_1, \ldots, a_n) = \delta_0(b_1, \ldots, b_p)?$$

The mystery of the invariance issue is this: how does the asymptotic geometry of the microstate spaces reflect properties of the generated von Neumann algebra of the n-tuple?

We showed in [5] that δ_0 possesses a fractal geometric description in terms of uniform packings. Encouraged by this result we use microstates to develop fractal geometric quantities for an n-tuple of selfadjoint operators in a tracial von Neumann algebra. For such an n-tuple $z_1, \ldots, z_n$ we define several numerical measurements of their microstate spaces, the most notable being the free Hausdorff dimension of $z_1, \ldots, z_n$. We denote this quantity by $\mathbb{H}(z_1, \ldots, z_n)$. As in the classical case we have that $\mathbb{H}(z_1, \ldots, z_n) \leqslant \delta_0(z_1, \ldots, z_n)$. For each $\alpha \in \mathbb{R}_+$ we define an α-free Hausdorff entropy for $z_1, \ldots, z_n$ which we denote by $\mathbb{H}^\alpha(z_1, \ldots, z_n)$. Hausdorff n-measure is a constant multiple of Lebesgue measure and in our setting we have an analogous statement: $\mathbb{H}^n(z_1, \ldots, z_n) = \chi(z_1, \ldots, z_n) + \frac{n}{2} \log\left(\frac{2n}{\pi e}\right)$. A modified version of $\mathbb{H}$ denoted by $\overline{\mathbb{H}}$ turns out to be an algebraic invariant. We compute the free Hausdorff dimension of the n-tuple when it generates a finite dimensional algebra or when $n = 1$. In both cases the free Hausdorff and free entropy dimensions agree. Using the computations for a single selfadjoint, we show that if $\mathbb{H}(z_1, \ldots, z_n) < 1$, then $\{z_1, \ldots, z_n\}''$ has a minimal projection. We view this as a microstates analogue of the classical fact that a metric space with Hausdorff dimension strictly less than 1 must be totally disconnected. Finally we show $\mathbb{H}$ becomes additive in the presence of freeness for the two aforementioned n-tuples of random variables.

Our motivation in developing fractal dimensions for microstate spaces is twofold. Firstly, having other metric measurements for them may eventually shed light on the connections between their asymptotic geometry and the structure of the generated von Neumann algebras. Secondly, it provides another conceptual framework for the micro state theory.

Section 2 is a list of notation. Section 3 reviews the definition of classical Hausdorff dimension, then presents the free Hausdorff dimension and entropy of an n-tuple and some of its basic properties. The section concludes with free packing and Minkowski-like entropies. Section 4 introduces $\overline{\mathbb{H}}$, the modified version of $\mathbb{H}$, and shows that $\overline{\mathbb{H}}$ is an algebraic invariant. Section 5 deals with the free Hausdorff dimension of finite-dimensional algebras. Section 6 deals with the free Hausdorff dimension of single