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Journal of Functional Analysis 226 (2005) 230–255
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Exponential ergodicity for stochastic Burgers and 2D Navier–Stokes equations☆

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Received 1 September 2004; accepted 28 December 2004
Communicated by Paul Malliavin
Available online 17 March 2005

Abstract

It is shown that transition measures of the stochastic Navier–Stokes equation in $2D$ converge exponentially fast to the corresponding invariant measures in the distance of total variation. As a corollary we obtain the existence of spectral gap for a related semigroup obtained by a sort of ground state transformation. Analogous results are proved for the stochastic Burgers equation.

Keywords: Navier–Stokes equation; Burgers equation; Additive noise; Invariant measure; Uniform exponential ergodicity; Spectral gap

1. Introduction

In this paper we study ergodic behaviour of two important equations arising in Statistical Physics: the stochastic Burgers equation and the stochastic Navier–Stokes equation in $2D$. In both cases we assume that the random forcing is correlated in space and white in time. The problem of ergodicity and the rate of convergence to invariant measure in various norms for those two equations was an object of intense

☆ This work was partially supported by the ARC Discovery Grant DP0346406, the UNSW Faculty Research Grant PS05345 and the GAČR Grant 201/04/0750.
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doi:10.1016/j.jfa.2004.12.009

research in recent years. In the paper [15] the existence and uniqueness of invariant measure for the stochastic Navier–Stokes equation in $2D$ was proved in the case when the random force is sufficiently close to the space–time white noise. The exponential rate of convergence of transition measures to the invariant measure μ of the stochastic Navier–Stokes equation was proved for the first time in [1] for μ-almost every initial condition and subsequently for every square integrable initial condition in [24,27] (see also [10]). In all these papers various versions of coupling technique were applied to prove the convergence properties in metrics equivalent to the topology of weak convergence of measures (or an intermediate metric, cf. [27]). The coupling method proved also to be useful to handle random forces which are degenerated in space. Uniqueness of invariant measure for the Navier–Stokes equation perturbed by finite-dimensional Wiener process, has been proven in a recent paper [19], see also [18]. Interesting results on the regularity of the law of the finite dimensional projection of the solution at time $t > 0$ were obtained in [20]. Let us note also that similar result were obtained for the forcing consisting of a sequence of random excitations arising in discrete moments of time (random kicks), see e.g. [23].

In this work we continue the approach initiated in [15] assuming that the random force is sufficiently nondegenerate. In particular, we are using results of [3,9,11,12], where the strong Feller property and irreducibility have been proven in an appropriately chosen state space for particular cases of the stochastic Navier–Stokes equation and stochastic Burgers equation. Our main result may be described as follows. Let $\{u(t,\zeta) : t \geqslant 0, \zeta \in \mathcal{O}\}$ be a solution to either the stochastic Navier–Stokes equation (in which case $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$) or a solution to the stochastic Burgers equation (and then $\mathcal{O} = (0,1)$) and let μ be the corresponding invariant measure. Then for any initial distribution v of the $L^2(\mathcal{O})$-valued random variable $u(0,\cdot)$ the probability distribution $P_t^* v$ of the random variable $u(t,\cdot)$ enjoys the property

$$\|P_t^* v - \mu\|_{\mathrm{var}} \leqslant \|P_t^* v - \mu\|_V \leqslant C e^{-\beta t} \|v\|_V, \tag{1.1}$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the norm of total variation of measures and $\|v\|_V$ stands for the norm of total variation of the measure $V dv$ considered on $L^2(\mathcal{O})$. The function $V : L^2(\mathcal{O}) \to [1,\infty)$ in (1.1) is an appropriate Lyapunov function. A class of functions V for which (1.1) holds is also provided and shown to include $V(x) = 1 + |x|_{L^2}^p$ (for $p > 0$) and $V(x) = \exp(|x|_{L^2}^{2\alpha})$ (for $\alpha \in (0,1)$). Finally, we derive from (1.1) the spectral gap property of the V-transform (P_t^V)

$$P_t^V \phi(x) = V^{-1}(x)\mathbb{E}(V(u(t,\cdot))\phi(u(t,\cdot)))$$

of the semigroup (P_t). Namely, we show that for $\phi \in L^p(H, V\mu)$, $p \in (1,\infty)$

$$\int_{L^2} \left| P_t^V \phi - V^{-1} \int_{L^2} \phi V d\mu \right|^p V d\mu \leqslant C_p e^{-\beta t/p} \int_{L^2} |\phi|^p V d\mu. \tag{1.2}$$

Exponential convergence to the invariant measure in the distance of total variation and the spectral gap property (1.2) seem to be new for both equations studied in this paper. The main idea of the proof consists in verifying the V-uniform ergodicity for a