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A Liouville-type result for quasi-linear elliptic equations on complete Riemannian manifolds

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Abstract

We extend a Liouville-type result of D. G. Aronson and H. F. Weinberger and E.N. Dancer and Y. Du concerning solutions to the equation $\Delta_p u = b(x)f(u)$ to the case of a class of singular elliptic operators on Riemannian manifolds, which include the φ-Laplacian and are the natural generalization to manifolds of the operators studied by J. Serrin and collaborators in Euclidean setting. In the process, we obtain an a priori lower bound for positive solutions of the equation in consideration, which complements an upper bound previously obtained by the authors in the same context.

MSC: 58J05; 53C21

Keywords: A priori estimates; Maximum principles; Volume growth; Quasi-linear elliptic inequalities

0. Introduction

Let f be a continuous function on $\mathbb{R}$ satisfying the conditions

$$\text{(i) } f(0) = f(a) = 0, \ \text{(ii) } f(s) > 0 \text{ in } (0, a), \ \text{(iii) } f(s) < 0 \text{ in } (a, +\infty). \tag{0.1}$$

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In recent years, the study of the semilinear diffusion equation

$$u_t = \Delta u + f(u) \quad \text{on } [0, +\infty) \times \mathbb{R}^m, \tag{0.2}$$

which arises in population biology and chemical reaction theory, has attracted the attention of many researchers in the field, see [AW1] and [AW2] for references and background.

In [AW2], Aronson and Weinberger showed that if f is C^1 and

$$\liminf_{s \to 0+} \frac{f(s)}{s^{1+2/m}} > 0, \tag{0.3}$$

then a “hair trigger” effect takes place, and any non-identically zero solution $u(x, t)$ of (0.2) with values in $[0, a]$ is such that

$$\lim_{t \to +\infty} u(x, t) = a,$$

uniformly in $x \in \mathbb{R}^m$. Moreover, the exponent $1 + 2/m$ in (0.3) is sharp in the sense that the hair trigger effect fails if $1 + 2/m$ in (0.3) is replaced by any larger σ.

As a consequence of the hair trigger effect one deduces a Liouville result for the elliptic problem associated to (0.4), namely, any solution u of

$$\Delta u + f(u) = 0 \tag{0.4}$$

with values in $[0, a]$ is constant and identically equal to either 0 or a.

It should be noted that the assumption $f(s) < 0$ for $s > a$ implies that any non-negative, globally bounded solution u of (0.4) satisfies $0 \leqslant u \leqslant a$, and that if f is superlinear at $+\infty$, then any non-negative solution is in fact globally bounded (see [DM,PRS1]).

As for the sharpness of the exponent $1 + 2/m$ in (0.3) in order that this kind of Liouville-type result hold, it was shown by Dancer [D], that if $m > 2$ and $\sigma > m/(m-2)$ one can find a function $f \in C^1(\mathbb{R})$ satisfying (0.1) and $f(s) \geqslant cs^\sigma$ for $s \to 0+$, such that (0.4) has a positive solution u with $0 < u < a$ which tends to zero at infinity.

In subsequent work, Du and Guo, [DG], extended the investigation to the case of the p-Laplace operator, and conjectured that if $m > p$ then the sharp exponent should be given by Serrin's exponent $\sigma = m(p-1)/(m-p)$ (which reduces to $\sigma = m/(m-2)$ in the case of the Laplacian).

The conjecture has been recently established by Dancer and Du [DD], using results due to Bidaut-Veron and Pohozaev [BVP], and to Serrin and Zou [SZ]. For the sake of comparison, we report here their result.

Theorem (Dancer and Du [DD]). *Let f be continuous on $[0, +\infty)$ and locally quasi-monotone* (*in the sense that for any bounded interval $[s_1, s_2]$ contained in $[0, +\infty)$*