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Area inequality and Q_p norm[☆]

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Abstract

In this paper, we introduce the norm squares B_p for Q_p spaces on a hyperbolic Riemann surface R so that $B_p(f)=1$ for $0\leqslant p\leqslant\infty$ if R is the unit disk and f is the identity function, and prove the sharp inequality $B_p(f)\leqslant B_q(f)$ for $0\leqslant q<p\leqslant\infty$. The equality statement is also settled. This is a stronger version of the known nesting property: $AD(R)=Q_0(R)\subset Q_q(R)\subset Q_p(R)\subset Q_\infty(R)=\mathscr{CB}(R)$ for $0<q<p<\infty$, where $AD(R)$ and $\mathscr{CB}(R)$ are the Dirichlet space and Bloch-type space, respectively.

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1. Introduction

Let R be a hyperbolic Riemann surface, and let f be a function defined and analytic on R. The Dirichlet integral $D(f)$ is defined by

$$D(f)=\frac{1}{\pi}\int\int_R |f'(z)|^2\,dx\,dy$$

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and the Dirichlet space $AD(R)$ is defined as the space which consists of all analytic functions on R with finite Dirichlet integrals. Let

$$B(f) = \sup_{a \in R} \frac{2}{\pi} \int\int_R |f'(z)|^2 g(z,a)\, dx\, dy < \infty,$$

where $g(z,a)$ denotes a Green's function of R with logarithmic singularity at a. The square root $B(f)^{1/2}$ is called the *BMO* norm of *f*. The space $BMOA(R)$ is defined as the class of all analytic functions whose *BMO* norms are finite.

The notion of spaces Q_p was first considered for analytic functions defined in the unit disk Δ of the complex plane (cf. [7,9,10,14,16]) and, later, generalized to hyperbolic Riemann surfaces (cf. [8,6]). A good overall review of Q_p spaces can be found in the monograph [15]. For $p \geqslant 0$, let

$$B_p(f) = \sup_{a \in R} \frac{2^p}{\pi \Gamma(p+1)} \int\int_R |f'(z)|^2 g^p(z,a)\, dx\, dy.$$

Now $B_p(f)^{1/2}$ is called the Q_p norm of *f*. We denote by $Q_p(R)$ the space of functions *f* analytic on R for which $B_p(f) < \infty$. Note that $B_0(f) = D(f)$, $B_1(f) = B(f)$, $Q_0(R) = AD(R)$ and $Q_1(R) = BMOA(R)$.

The inclusion $AD(R) \subset BMOA(R)$ was proved first by Metzger [13, Theorem 1] and later by Kobayashi [11, Corollary 2], in a different way, by showing a sharp inequality $B(f) \leqslant D(f)$. Actually, Kobayashi proved that

$$\int\int_R |f'(z)|^2 g(z,a)\, dx\, dy \leqslant \frac{1}{2} \int\int_R |f'(z)|^2\, dx\, dy \tag{1.1}$$

holds for any $a \in R$. In [6], the nesting property of these function spaces, that is, $Q_q \subset Q_p$ for $p > q \geqslant 0$, was established and the relation between Q_p spaces and Bloch-type spaces was considered. However, they did not give any other explicit relation among norm squares $B_p(f)$, $0 \leqslant p < \infty$, similar to what Kobayashi gave.

The main result of this paper is a generalization of the above result of Kobayashi: for any $p > q \geqslant 0$ and any $a \in R$, we have

$$\int\int_R |f'(z)|^2 g^p(z,a)\, dx\, dy \leqslant \frac{2^{q-p}\Gamma(p+1)}{\Gamma(q+1)} \int\int_R |f'(z)|^2 g^q(z,a)\, dx\, dy \tag{1.2}$$

and, consequently, $B_p(f) \leqslant B_q(f)$. Similarly to Kobayashi's result, our generalization is also sharp. Let $R_t = \{z \in R : g(z,a) > t\}$ and

$$\phi(t) = \int\int_{R_t} |f'(z)|^2\, dx\, dy,$$