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Decomposition rank and absorbing extensions of type I algebras

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Abstract

We prove the following: Let A and B be separable C^*-algebras. Suppose that B is a type I C^*-algebra such that

(i) B has only infinite dimensional irreducible $*$-representations, and
(ii) B has finite decomposition rank.

If

$$0 \to B \to C \to A \to 0$$

is a unital homogeneous exact sequence with Busby invariant τ, then the extension τ is absorbing.

In the case of infinite decomposition rank, we provide a counterexample. Specifically, we construct a unital, homogeneous, split exact sequence of the form

$$0 \to C(Z) \otimes \mathscr{K} \to C \to \mathbb{C} \to 0$$

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which is not absorbing. In this example, Z is an infinite-dimensional, compact, second countable topological space. This gives a counterexample to the natural infinite-dimensional generalization of the result of Pimsner, Popa and Voiculescu.

Keywords: Absorbing extensions; C*-algebras

1. Introduction

In view of the interesting results of Villadsen [17] and of Rørdam [15], it seems timely to make a few remarks about absorbing extensions involving C^*-algebras with infinite base spaces. Recall that Pimsner et al. [13] showed that a family of extensions satisfying the hypothesis of Voiculescu's absorption theorem [18] at every point would, if the family satisfied a certain condition called homogeneity, be absorbing. We generalize this result in several directions simultaneously. Replacing the compact operators by a reasonably general class of type I C^*-algebras, we firstly implicitly prove a counterpart of Voiculescu's absorption theorem for this class of type I algebra, using our earlier results [5], and then, generalizing Pimsner, Popa, and Voiculescu's definition of homogeneity, we prove a counterpart of their absorption theorem for families. There appears to be a fundamental obstacle if the spectrum is infinite-dimensional. Using Kirchberg and Winter's [8] notion of decomposition rank to measure the noncommutative dimension of the algebra, we establish an absorption theorem for the case of finite decomposition rank, and give an example showing that the case of infinite decomposition rank is fundamentally different. This example uses many of the techniques (partially due to [17]) used to establish Rørdam's celebrated example of a C^*-algebra containing both a finite and an infinite projection [15] . We note that extensions of type I C^*-algebras are of general interest, since, if G is either a simply connected nilpotent Lie group or a connected semisimple Lie group, then the group C^*-algebra $C^*(G)$ is type I, and the structure theory of $C^*(G)$ is still not complete.

2. The case of finite decomposition rank

The theorem to be proven is: [1]

Theorem 2.1. *Let A and B be separable C^*-algebras. Suppose that B is a type I C^*-algebra such that*

(i) *B has only infinite dimensional irreducible $*$-representations, and*
(ii) *B has finite decomposition rank.*

[1] Throughout this paper, the term representation means a $*$-representation, and the term ideal always means a two-sided closed ideal, generally with respect to the norm topology. We sometime use other topologies, which are then indicated by suitable adjectives.