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About a construction and some analysis of time inhomogeneous diffusions on monotonely moving domains

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Abstract

We construct and analyze in a very general way time inhomogeneous (possibly also degenerate or reflected) diffusions in monotonely moving domains $E \subset \mathbb{R} \times \mathbb{R}^d$, i.e. if $E_t := \{x \in \mathbb{R}^d | (t,x) \in E\}$, $t \in \mathbb{R}$, then either $E_s \subset E_t$, $\forall s \leqslant t$, or $E_s \supset E_t$, $\forall s \leqslant t$, $s,t \in \mathbb{R}$. Our major tool is a further developed $L^2(E,m)$-analysis with well chosen reference measure m. Among few examples of completely different kinds, such as e.g. singular diffusions with reflection on moving Lipschitz domains in $\mathbb{R}^d$, non-conservative and exponential time scale diffusions, degenerate time inhomogeneous diffusions, we present an application to what we name skew Bessel process on γ. Here γ is either a monotonic function or a continuous Sobolev function. These diffusions form a natural generalization of the classical Bessel processes and skew Brownian motions, where the local time refers to the constant function $\gamma \equiv 0$.

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0. Introduction

The theory of time dependent Dirichlet forms has been inspired from [11,12], and recently by [13]. We present here an independent, self-contained and more enclosing finite dimensional analysis, being integrated in the general theory of generalized Dirichlet forms. The present exposition can also be seen as a completion to [17], and vice versa.

We are concerned with the development of a new technique for the general treatment of time dependent diffusions on monotonely moving domains $E \subset \mathbb{R} \times \mathbb{R}^d$ with generator

$$LF(s,x) = \sum_{i,j=1}^{d} a_{ij}(s,x)\partial_i\partial_j F(s,x) + \sum_{i=1}^{d} b_i(s,x)\partial_i F(s,x) - c(s,x)F(s,x) + d(s)\partial_t F(s,x),$$

where the (symmetric) diffusion matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$ may be degenerate, b_i discontinuous, non locally bounded, c is positive and bounded, and L may be regarded together with some boundary conditions. At the present stage we give as examples $d \equiv 1$, and $d(s) = sd$, where d is a positive constant.

Actually L may be examined with absorbing boundary conditions on some "freely" chosen $J \subset \partial E$, and with reflecting boundary conditions on the complement $\partial E \setminus J$. In order to not overload the exposition of this paper we concentrate exclusively on reflecting boundary conditions in case of existent boundary (cf. Remark 1.10).

For $s \in \mathbb{R}$ let $E_s := \{x \in \mathbb{R}^d | (s,x) \in E\}$. A monotonely moving domain $E \subset \mathbb{R} \times \mathbb{R}^d$ is by our definition a closed domain which satisfies either $E_s \subset E_t$, or $E_t \subset E_s$, for all $s \leqslant t$ in some time interval of $\mathbb{R}$. If $E_s \equiv E_t$, we are concerned with "constantly moving domains". These are cylindrical domains such as e.g. $\mathbb{R} \times \mathbb{R}^d$, or more generally such as e.g. $\mathbb{R} \times D$, $D \subset \mathbb{R}^d$. For simplicity we assume that the time interval T_I is given by $\mathbb{R}$, or by $\mathbb{R}^+ := [0, \infty)$. We can write $E = \bigcup_{s \in T_I} \{s\} \times E_s$.

If

$$b_i = \frac{1}{2} \sum_{j=1}^{d} \left(\partial_j a_{ij} + a_{ij} \frac{\partial_j \rho}{\rho} \right)$$

with sufficiently regular ρ, $A = (a_{ij})_{1 \leqslant i,j \leqslant d}$, and ∂E, an integration by parts w.r.t. the "well-chosen" measure $dm = \rho(s,x)\,dx\,ds$ gives

$$-\int_E LF \cdot G\,dm = \frac{1}{2}\int_E \langle A\nabla F, \nabla G\rangle\,dm + \int_E cFG\,dm - \int_E \Lambda F \cdot G\,dm, \qquad (1)$$