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Rational hyperholomorphic functions in $\mathbb{R}^4$

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Abstract

We introduce the notion of rationality for hyperholomorphic functions (functions in the kernel of the Cauchy–Fueter operator). Following the case of one complex variable, we give three equivalent definitions: the first in terms of Cauchy–Kovalevskaya quotients of polynomials, the second in terms of realizations and the third in terms of backward-shift invariance. Also introduced and studied are the counterparts of the Arveson space and Blaschke factors.

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1. Introduction

It is well known that functions holomorphic in a domain $\Omega \subset \mathbb{C}$ are exactly the elements of the kernel of the Cauchy–Riemann differential operator

$$\overline{\partial} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

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restricted to Ω. A polynomial in x and y is holomorphic if, and only if, it is a polynomial in the complex variable $z = x + iy$, and rational holomorphic functions are quotients of polynomials.

Holomorphic functions of one complex variable have a natural generalization to the quaternionic setting when one replaces the Cauchy–Riemann operator by the Cauchy–Fueter operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{e_1}\frac{\partial}{\partial x_1} + \mathbf{e_2}\frac{\partial}{\partial x_2} + \mathbf{e_3}\frac{\partial}{\partial x_3}.$$

In this expression the x_j are real variables and the $\mathbf{e_j}$ are imaginary units of the skew-field $\mathbb{H}$ of quaternions (see Section 2 below for more details). Solutions of the equation $Df = 0$ are called left-hyperholomorphic functions (they are also called left-hyperanalytic, or left-monogenic, or regular, functions, see [18,13,23]). Right-hyperholomorphic functions are the solutions of the equation

$$fD = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}\mathbf{e_1} + \frac{\partial f}{\partial x_2}\mathbf{e_2} + \frac{\partial f}{\partial x_3}\mathbf{e_3} = 0.$$

When trying to generalize the notions of polynomial and rational functions to the hyperholomorphic setting, one encounters several obstructions. For instance, the quaternionic variable

$$x = x_0 + x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}$$

is not hyperholomorphic. Moreover, the point-wise product of two hyperholomorphic functions is not hyperholomorphic in general and the point-wise inverse of a non-vanishing hyperholomorphic function need not be hyperholomorphic.

For the polynomials these difficulties were overcome by Fueter, who introduced in [16] the symmetrized multi-powers of the three elementary functions

$$\zeta_1(x) = x_1 - \mathbf{e_1}x_0, \quad \zeta_2(x) = x_2 - \mathbf{e_2}x_0, \quad \text{and} \quad \zeta_3(x) = x_3 - \mathbf{e_3}x_0.$$

The polynomials thus obtained are known today as the Fueter polynomials. They are (both right and left) hyperholomorphic and appear in power series expansions of hyperholomorphic functions. In particular, a hyperholomorphic polynomial is a linear combination of the Fueter polynomials.

In this paper we introduce the notion of rational hyperholomorphic function. We obtain three equivalent characterizations: the first one in terms of quotients and products of polynomials, the second one in terms of realization and the last one in terms of backward-shift-invariance. These various notions need to be suitably defined in the hyperholomorphic setting. A key tool here is the Cauchy–Kovalevskaya product of hyperholomorphic functions.