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Journal of Functional Analysis 221 (2005) 150–166

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Noncommutative L^p structure encodes exactly Jordan structure

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Received 14 January 2004; received in revised form 3 August 2004; accepted 7 September 2004
Communicated by G. Pisier
Available online 5 November 2004

Abstract

We prove that for all $1 \leqslant p \leqslant \infty$, $p \neq 2$, the L^p spaces associated to two von Neumann algebras $\mathscr{M}$, $\mathscr{N}$ are isometrically isomorphic if and only if $\mathscr{M}$ and $\mathscr{N}$ are Jordan *-isomorphic. This follows from a noncommutative L^p Banach–Stone theorem: a specific decomposition for surjective isometries of noncommutative L^p spaces.

MSC: primary: 46L52; secondary: 46B04; 46L10

Keywords: von Neumann algebra; Noncommutative L^p space; Isometry; Jordan isomorphis

1. Introduction

In this paper we prove the following theorem.

Theorem 1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, and $1 \leqslant p \leqslant \infty$, $p \neq 2$. The following are equivalent*:
(1) *$\mathcal{M}$ and $\mathcal{N}$ are Jordan *-isomorphic*;
(2) *$L^p(\mathcal{M})$ and $L^p(\mathcal{N})$ are isometrically isomorphic as Banach spaces*.

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.09.003

$L^\infty(\mathcal{M})$ is to be understood as $\mathcal{M}$ itself, so for $p = \infty$ the statement follows from the classic article of Kadison [14] (see Theorems 2.1 and 2.2 below). One may view this paper as an L^p version of Kadison's results.

The implication (1) → (2) is a direct application of modular theory and interpolation, only requiring us to go a little further down well-traveled paths. The more interesting part is to show that (2) → (1). In case the surjective isometry is *-preserving and the algebras are σ-finite, this was proved by Watanabe [32]. When both algebras are semifinite, this follows from a structure theorem for L^p isometries (even non-surjective) due to Yeadon [37,28]; recently Yeadon's theorem was extended in [13] to classify isometries for which only the initial algebra is assumed semifinite. In common with these papers, our proof relies crucially on the equality condition in the noncommutative Clarkson inequality. But we do not make use of any of these papers' results, and type considerations play no role in our argument (although abelian summands require a little extra care). We actually determine the structure of the surjective isometry, as follows.

Theorem 1.2 (*Noncommutative L^p Banach–Stone theorem*). *Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be a surjective isometry, where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras and $1 < p < \infty$, $p \neq 2$. Then there are a surjective Jordan *-isomorphism $J : \mathcal{M} \to \mathcal{N}$ and a unitary $w \in \mathcal{N}$ such that*

$$T(\varphi^{1/p}) = w(\varphi \circ J^{-1})^{1/p}, \qquad \forall \varphi \in \mathcal{M}_*^+. \tag{1.1}$$

Here $\varphi^{1/p}$ is the generic positive element of $L^p(\mathcal{M})$; we will explain this notation. Since any L^p element is a linear combination of four positive ones, (1.1) completely determines T. The extensions to $0 < p \leqslant 1$ of Theorems 1.1 and 1.2 are true but not proved in this paper—see Remark 2 of Section 5, and [26].

A version of Theorem 1.2 was shown by Watanabe [35] under the assumptions that T is *-preserving and $\mathcal{M}$ has a certain extension property. Our method here is different: we focus on the subspaces $q_1 L^p(\mathcal{M}) q_2$, where q_1, q_2 are projections in $\mathcal{M}$. These subspaces, called corners, are a sort of "two-dimensional" analogue of the projection bands in classical L^p spaces. It turns out that T takes corners to corners, preserving both orthogonality (in the sense defined below) and the semi-inner product. From this we deduce the existence of an orthoisomorphism between the projection lattices of $\mathcal{M}$ and $\mathcal{N}$. Extending the orthoisomorphism produces a Jordan *-isomorphism, and an intertwining relation finally implies that T has form (1.1).

Theorem 1.2 evidently suggests the larger challenge of classifying all L^p isometries. While this is still open in general, we mention that the author has recently written an article [25] which obtains several new results, including a solution which is valid under a mild (perhaps vacuous?) hypothesis on the initial algebra. Also the paper [13] completely determines the structure of 2-isometries between L^p spaces. Although there is some overlap in the setup of these problems, we believe that the surjective case merits a separate exposition, being of independent interest and admitting a distinct technique and solution. There is no overlap at all—in fact, an interesting contrast—