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The short-cut test

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Abstract

The short-cut test detects existence and uniqueness of "Laplacians" on finitely ramified, graph-directed fractals. Previous results by Sabot, Nussbaum and the author are improved and extended. It opens up the way for further studies because it combines well established spectral, dynamical and analytic techniques. Its algorithmic and recursive structure is designed to provide computable and flexible criteria.

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1. Introduction and results

Fractal sets are physically relevant because they resemble porous media. On them one would like to study diffusions or prototypically heat conduction. To construct it one has to find a "Laplace operator" Δ on the fractal F which then defines a heat semigroup via $P_t = e^{\Delta t}$. We will equivalently look for the Dirichlet form $\mathcal{D}(f, g) := \langle -\Delta f, g\rangle_{L^2(F,\mu)}$ on the fractal equipped with its normalized Hausdorff measure μ. The existence and uniqueness of such Dirichlet forms, under additional symmetry, regularity and scaling assumptions, on so-called finitely ramified, graph-directed fractals is the

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topic of this article. "Finitely ramified" roughly means that one can ruin the connection of F by removing only finitely many points. "Graph-directed" allows us to mix a finite number of building blocks in the construction of the fractal. Unlike Lindstrøm's existence results for the relatively small class of "nested fractals" no general existence or uniqueness result is proven. Instead a tool box, the short-cut test, is presented which should enable the reader to answer the existence or uniqueness question in a flexible algorithmic way. In the examples tested so far it was a real short cut as compared to previously available techniques. Especially, Sabot's results in [32] on existence and uniqueness are improved. They are in turn an improvement of Barlow's considerations in [3].

The article covers a wide class of new examples, because previous articles mainly dealt with constructions based on a single building block. Only two exceptions came to the knowledge of the author, the diamond and the Hany fractal [14,24]. Fractals which are not finitely ramified (infinitely ramified), like the Sierpinski carpet studied in [4], are not covered by this article.

On finitely ramified fractals the existence and uniqueness of a Dirichlet form $\mathcal{E}$ is known to be equivalent to the existence and uniqueness of a discrete self-similar Dirichlet form on a finite skeleton of the fractal, as described by Lindstrøm. Such a Dirichlet form is the eigenvector, located in the interior of a cone P, of a nonlinear renormalization map Λ. Up to nonlinearity this is the set up of classical Perron–Frobenius theory. We will use a nonlinear version of it known as Hilbert's projective metric on cones. The geometric aspect of this theory is a "weakly" Gromov hyperbolic metric space with Hilbert's metric h on P in the sense of [12], the spectral side is an interval calculus imitating classical results on the spectral radius due to Collatz, Wielandt and Nussbaum, and the dynamical side is the iteration of the h-nonexpansive map Λ also studied by Nussbaum. The nonexpansiveness implies an aggregated action of Λ on so called parts. It mimics the closed classes of states of a Markov chain and is a way to analyze the "irreducibility" of Λ. The aggregated action decomposes the original eigenvalue problem into a small collection of lower dimensional subproblems of the same type. This is the "cut" aspect of the short-cut test. On irreducible subproblems we have to rely on nonlinearity. An efficient way to study the nonlinear aspects of the dynamics of Λ are Gâteaux derivatives at the boundary of P. This leads to monotone convergence problems in which infinite "conductances" appear. This "short circuiting of electrical networks" is the "shorting" aspect of the short-cut test. Several of the above techniques only use qualitative properties of Λ and are, therefore, also of general interest.

Surprisingly, at least to the author, these techniques turned out to be closely related to Sabot's arguments in [32]. As compared to his results the main differences are: graph-directed instead of single block fractals are considered, his very restrictive Assumption H is completely removed, "existence without uniqueness" is no more a blind spot, and further tests can be developed because well established techniques are used. Technically speaking, Proposition 24(ii) allows us to avoid Sabot's Assumption H. This statement in turn is a consequence of the geometric Lemma 6 and dynamic Proposition 15. The present article suggests four tests, the function test, Sabot's test, the eigenvalue test and Nussbaum's test. They can be freely combined in a recursive and algorithmic