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Sum rules for Jacobi matrices and divergent Lieb–Thirring sums

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Abstract

Let E_j be the eigenvalues outside $[-2, 2]$ of a Jacobi matrix with $a_n - 1 \in \ell^2$ and $b_n \to 0$, and μ' the density of the a.c. part of the spectral measure for the vector δ_1. We show that if $b_n \notin \ell^4$, $b_{n+1} - b_n \in \ell^2$, then

$$\sum_j (|E_j| - 2)^{5/2} = \infty$$

and if $b_n \in \ell^4$, $b_{n+1} - b_n \notin \ell^2$, then

$$\int_{-2}^{2} \ln(\mu'(x))(4 - x^2)^{3/2}\, dx = -\infty.$$

We also show that if $a_n - 1, b_n \in \ell^3$, then the above integral is finite if and only if $a_{n+1} - a_n, b_{n+1} - b_n \in \ell^2$. We prove these and other results by deriving sum rules in which the a.c. part of the spectral measure and the eigenvalues appear on opposite sides of the equation.

Keywords: Jacobi matrix; Sum rules; Lieb–Thirring sums

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1. Introduction

In the present paper we consider Jacobi matrices

$$J \equiv \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

with $a_n > 0$, $b_n \in \mathbb{R}$, and $a_n \to 1$, $b_n \to 0$. These are compact perturbations of the *free matrix* J_0 with $a_n \equiv 1$ and $b_n \equiv 0$. If only $a_n \equiv 1$, then J is the discrete half-line Schrödinger operator with the decaying potential b_n.

J is a self-adjoint operator acting on $\ell^2(\{1,2,\dots\})$. We denote by μ the spectral measure of the (cyclic for J) vector δ_1 and by μ' the density of its a.c. part. For J_0, the measure μ_0 is absolutely continuous with $\mu_0'(x) = (2\pi)^{-1}\sqrt{4-x^2}\chi_{[-2,2]}(x)$, and so by Weyl's theorem, $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(J_0) = [-2,2]$. Hence, outside $[-2,2]$ spectrum of J consists only of eigenvalues (of multiplicity 1), with ± 2 the only possible accumulation points. We will denote the negative ones $E_1, E_3, \dots$ and the positive ones $E_2, E_4, \dots$, with the convention that $E_{2j-1} \equiv -2$ ($E_{2j} \equiv 2$) if J has fewer than j eigenvalues below -2 (above 2).

We let $\partial a_n \equiv a_{n+1} - a_n$, $\partial b_n \equiv b_{n+1} - b_n$, and define

$$r_n \equiv b_n^4 - 2(\partial b_n)^2 - 8(\partial a_n)^2 + 4(a_n^2 - 1)(b_n^2 + b_n b_{n+1} + b_{n+1}^2).$$

The following are our main results.

Theorem 1. *Assume that $a_n - 1 \in \ell^3$ and $b_n \to 0$.*

(i) *If $\sum_{n=1}^\infty r_n = \infty$ or does not exist, then $\sum_{j=1}^\infty (|E_j| - 2)^{5/2} = \infty$.*

(ii) *If $\sum_{n=1}^\infty r_n = -\infty$ or does not exist, then $\int_{-2}^2 \ln(\mu'(x))(4-x^2)^{3/2}\,dx = -\infty$.*

Remark. One can actually dispense with the assumption $a_n - 1 \in \ell^3$, but the corresponding r_n is less transparent (it is the diagonal element of the matrix $P_w(J)$ from the proof of Theorem 1).

Corollary 2. *Assume that $a_n - 1 \in \ell^2$ and $b_n \to 0$.*

(i) *If $b_n \notin \ell^4$ and $\partial b_n \in \ell^2$, then $\sum_{j=1}^\infty (|E_j| - 2)^{5/2} = \infty$.*

(ii) *If $b_n \in \ell^4$ and $\partial b_n \notin \ell^2$, then $\int_{-2}^2 \ln(\mu'(x))(4-x^2)^{3/2}\,dx = -\infty$.*

Proof. Since $a_n - 1 \in \ell^2$, we have $\partial a_n \in \ell^2$. Also,

$$|4(a_n^2 - 1)(b_n^2 + b_n b_{n+1} + b_{n+1}^2)| \leqslant 72(a_n^2-1)^2 + \tfrac{1}{4}(b_n^4 + b_{n+1}^4)$$

and $a_n^2 - 1 \in \ell^2$, so the result follows from Theorem 1. $\square$