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Journal of Functional Analysis 218 (2005) 255–292
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function

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Received 6 January 2004; accepted 15 April 2004

Available online 9 June 2004

Communicated by J. Bourgain

Abstract

In this paper we consider the discrete one-dimensional Schrödinger operator with quasi-periodic potential $v_n = \lambda v(x + n\omega)$. We assume that the frequency ω satisfies a strong Diophantine condition and that the function v belongs to a Gevrey class, and it satisfies a transversality condition. Under these assumptions we prove—in the perturbative regime—that for large disorder λ and for most frequencies ω the operator satisfies Anderson localization. Moreover, we show that the associated Lyapunov exponent is positive for all energies, and that the Lyapunov exponent and the integrated density of states are continuous functions with a certain modulus of continuity. We also prove a partial nonperturbative result assuming that the function v belongs to some particular Gevrey classes.

MSC: 81Q10; 47B39; 37D25; 82B44

Keywords: Discrete quasi-periodic Schrödinger operator; Anderson localization; Lyapunov exponent; Integrated density of states

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[1]The author was supported in part by NSF Grant DMS-9970469 and by NSF Grant DMS-9985572.

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.04.009

1. Introduction and statements

The discrete one-dimensional Schrödinger operator with quasi-periodic potential is the selfadjoint, bounded operator $H(x) = H_{\omega,\lambda}(x)$ on $l_2(\mathbb{Z})$ defined by

$$H_{\omega,\lambda}(x) := -\Delta + \lambda v(x + n\omega)\delta_{n,n'}, \tag{1.1}$$

where Δ is the discrete (lattice) Laplacian on $l_2(\mathbb{Z})$:

$$(\Delta u)_n := u_{n+1} + u_{n-1} - 2u_n. \tag{1.2}$$

In (1.1), v is a real-valued function on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, that is, a real-valued 2π-periodic function on $\mathbb{R}$, x is a parameter on $\mathbb{T}$, ω is an irrational frequency and λ is a real number called the disorder of the system.

We may assume the following on the data:

- (Strong) Diophantine condition on the frequency : $\omega \in DC_\kappa \subset \mathbb{T}$ for some $\kappa > 0$. That is,

$$\text{dist}(k\omega, 2\pi\mathbb{Z}) =: ||k\omega|| > \kappa \cdot \frac{1}{|k|(\log(1 + |k|))^3} \ \forall\, k \in \mathbb{Z} \backslash \{0\} \tag{1.3}$$

  Notice that $\text{mes}[\mathbb{T} \backslash DC_\kappa] \lesssim \kappa$.
- Gevrey-class regularity on the function: v is a smooth function which belongs to a Gevrey class $G^s(\mathbb{T})$ for some $s > 1$. That is,

$$\sup_{x \in \mathbb{T}} |\partial^m v(x)| \leqslant MK^m (m!)^s \quad \forall\, m \geqslant 0 \tag{1.4}$$

  for some constants $M, K > 0$.

  This condition is equivalent (see [Ka, Chapter V.2]) to the following exponential-type decay of the Fourier coefficients of v:

$$|\hat{v}(k)| \leqslant Me^{-\rho|k|^{1/s}} \quad \forall\, k \in \mathbb{Z} \tag{1.5}$$

  for some constants $M, \rho > 0$, where

$$v(x) = \sum_{k \in \mathbb{Z}} \hat{v}(k) e^{ikx} \tag{1.6}$$

  We will use (1.5) instead of (1.4).
- Transversality condition on the function: v is not flat at any point. That is:

$$\forall\, x \in \mathbb{T}\ \exists\, m \geqslant 1 \quad \text{so that } \partial^m v(x) \neq 0. \tag{1.7}$$

Notice from (1.4) or (1.5) with $s = 1$ that the Gevrey class $G^1(\mathbb{T})$ is the class of analytic functions on $\mathbb{T}$. The transversality condition (1.7) on a function