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Journal of Functional Analysis 218 (2005) 445–474
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Quasiregular representations of the infinite-dimensional Borel group

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Received 10 February 2004; accepted 20 March 2004

Communicated by Paul Malliavin

Abstract

The notion of quasiregular (Representation of Lie groups, Nauka, Moscow, 1983) or geometric (Grundlehren der Mathematischen Wissenschaften, Band 220, Springer, Berlin, New York, 1976; Encyclopaedia of Mathematical Science, Vol. 22, Springer, Berlin, 1994, pp. 1–156) representation is well known for locally compact groups. In the present work an analog of the quasiregular representation for the solvable infinite-dimensional Borel group $G = Bor_0^{\mathbb{N}}$ is constructed and a criterion of irreducibility of the constructed representations is presented. This construction uses G-quasi-invariant Gaussian measures on some G-spaces X and extends the method used in Kosyak (Funktsional. Anal. i Priložhen 37 (2003) 78–81) for the construction of the quasiregular representations as applied to the nilpotent infinite-dimensional group $B_0^{\mathbb{N}}$.

MSC: 22E65; 28C20; 43A80; 58D20

Keywords: Infinite-dimensional groups; Borel group; Regular representations; Quasiregular (geometric) representations; Irreducibility; Quasi-invariant measures; Ergodic measures; Ismagilov conjecture; Solvable group

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.03.009

1. Introduction

1.1. The setting and the main results

With any action $\alpha : G \to Aut(X)$ ($Aut(\cdot)$ denoting the group of all measurable automorphisms) of a group G on a G-space X (i.e. a space on which G acts) and G-quasi-invariant measure μ on X one can associate a unitary representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$, of the group G by the formula $(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), f \in L^2(X,\mu)$. Let us set $\alpha(G) = \{\alpha_t \in Aut(X) | t \in G\}$. Let $\alpha(G)'$ be centralizer of the subgroup $\alpha(G)$ in $Aut(X)$: $\alpha(G)' = \{g \in Aut(X) | \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e \forall t \in G\}$.

Conjecture 1 (Kosyak [27,28]). *The representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$ is irreducible if and only if*

(1) $\mu^g \perp \mu \forall g \in \alpha(G)' \backslash \{e\}$, (*where* $\perp$ *stands for singular*),
(2) *the measure* μ *is* G*-ergodic.*

We say that a measure μ is G-ergodic if $f(\alpha_t(x)) = f(x) \forall t \in G$ implies $f(x) = const$ for all functions $f \in L^1(X,\mu)$.

In this paper we shall prove Conjecture 1 in the case where G is the infinite-dimensional group, namely the Borel group $G = Bor_0^{\mathbb{N}}$, the space $X = X^m$ being the set of left cosets $G_m \backslash Bor^{\mathbb{N}}$, G_m suitable subgroups of the group $Bor^{\mathbb{N}}$ and μ any Gaussian product-measures on X^m. See below for explanation of the concepts used here.

1.2. Regular and quasiregular representations of locally compact groups

Let G be a locally compact group. The *right* ρ (respectively *left* λ) *regular representation* of the group G is a particular case of the representation $\pi^{\alpha,\mu,X}$ with the space $X = G$, the action α being the right action $\alpha = R$ (respectively the left action $\alpha = L$), and the measure μ being the right invariant Haar measure on the group G (see for example [9,21,22,45]).

A *quasiregular representation* of a locally compact group G is also a particular case of the representation $\pi^{\alpha,\mu,X}$ (see for example [45, p. 27]) with the space $X = H\backslash G$, the action α being the right action of the group G on the space X and the measure μ being some quasi-invariant measure on the space X (this measure is unique up to a scalar multiple). In [21,22] this representation is also called *geometric representation*.

1.3. An analog of the regular and quasiregular representations of infinite-dimensional groups and the Ismagilov conjecture

The work of Gel'fand played a decisive role in representation theory in general and in representation theory of infinite-dimensional groups, in particular see [11–13].