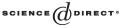


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On operator-valued spherical functions

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Abstract

We consider the equation

$$\int_{K} \Phi(x+k\cdot y) \, dk = \Phi(x)\Phi(y), \quad x, y \in G,$$
(1)

in which a compact group *K* with normalized Haar measure *dk* acts on a locally compact abelian group (G, +). Let \mathcal{H} be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the bounded operators on \mathcal{H} . Let $\Phi: G \to \mathcal{B}(\mathcal{H})$ any bounded solution of (0.1) with $\Phi(0) = I$:

(1) Assume *G* satisfies the second axiom of countability. If Φ is weakly continuous and takes its values in the normal operators, then $\Phi(x) = \int_K U(k \cdot x) dk$, $x \in G$, where *U* is a strongly continuous unitary representation of *G* on \mathcal{H} .

(2) Assuming G discrete, K finite and the map $x \mapsto x - k \cdot x$ of G into G surjective for each $k \in K \setminus \{I\}$, there exists an equivalent inner product on \mathcal{H} , such that $\Phi(x)$ for each $x \in G$ is a normal operator with respect to it.

Conditions (1) and (2) are partial generalizations of results by Chojnacki on the cosine equation.

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1. Introduction

Let (G, +) be an abelian group with neutral element 0. The *cosine equation*, also called d'Alembert's equation, on G is the equation

$$\frac{f(x+y) + f(x-y)}{2} = f(x)f(y), \quad x, y \in G,$$
(1.1)

where $f: G \to \mathbb{C}$ is the unknown.

The present paper deals with an extension of (1.1), both with respect to the form of (1.1) where a transformation group will enter, and to the range of f which will be in the bounded operators on a Hilbert space. The non-zero solutions of (1.1) are the functions of the form $f(x) = (\gamma(x) + \gamma(-x))/2$, $x \in G$, where γ is a homomorphism of G into the multiplicative group \mathbb{C}^* of non-zero complex numbers [19, Theorem 2]. If the solution is bounded, then γ is a homomorphism of G into the multiplicative group \mathbb{T} of complex numbers of modulus 1. We shall generalize this last fact.

Let *G* be an abelian topological group and let *K* be a compact topological transformation group of *G*, acting by automorphisms of *G*. Writing the action by $k \in K$ on $x \in G$ as $k \cdot x$ and letting *dk* denote the normalized Haar measure on *K* a generalization of the cosine equation (1.1) is

$$\int_{K} f(x+k \cdot y) dk = f(x)f(y), \quad x, y \in G,$$
(1.2)

where $f \in C(G)$ is the unknown. Eq. (1.2) is studied in the theory of group representations, being the relation defining *K*-spherical functions (for the terminology see [3, p. 88]).

Here are some examples: The Cauchy equation f(x + y) = f(x)f(y) has $K = \{I\}$. The cosine equation (1.1) is the case of $K = \mathbb{Z}_2 = \{\pm 1\}$, where the action by $-1 \in \mathbb{Z}_2$ on *G* is the group inversion. However, in our set up this action can be any involutive automorphism of $\sigma : G \to G$. The cosine equation then becomes

$$\frac{f(x+y) + f(x+\sigma(y))}{2} = f(x)f(y), \quad x, y \in G.$$
 (1.3)

The non-zero solutions of (1.3) are the functions of the form $f = (\gamma + \gamma \circ \sigma)/2$, where γ is a homomorphism of G into \mathbb{C}^* [2, Theorem 3, 29].

Another example of K is $\mathbb{Z}_N = \{\omega^n \mid n = 0, ..., N-1\}$, where $\omega = \exp(2\pi i/N)$, acting on $\mathbb{R}^2 = \mathbb{C}$ by multiplication. A further one is O(n) acting on \mathbb{R}^n by rotations.

The first part of the following general Theorem 1.1 is due to Shin'ya [28, Corollary 3.12], and the second part to Chojnacki [6, Theorem 1.1].

Theorem 1.1. Let G be a locally compact abelian Hausdorff topological group and let K be a compact topological transformation group of G, acting by automorphisms of G.

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