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On operator-valued spherical functions

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Abstract

We consider the equation

$$\int_K \Phi(x + k \cdot y)\, dk = \Phi(x)\Phi(y), \quad x, y \in G, \tag{1}$$

in which a compact group K with normalized Haar measure dk acts on a locally compact abelian group $(G, +)$. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the bounded operators on $\mathcal{H}$. Let $\Phi : G \to \mathcal{B}(\mathcal{H})$ any bounded solution of (0.1) with $\Phi(0) = I$:

(1) Assume G satisfies the second axiom of countability. If Φ is weakly continuous and takes its values in the normal operators, then $\Phi(x) = \int_K U(k \cdot x)\, dk$, $x \in G$, where U is a strongly continuous unitary representation of G on $\mathcal{H}$.

(2) Assuming G discrete, K finite and the map $x \mapsto x - k \cdot x$ of G into G surjective for each $k \in K \setminus \{I\}$, there exists an equivalent inner product on $\mathcal{H}$, such that $\Phi(x)$ for each $x \in G$ is a normal operator with respect to it.

Conditions (1) and (2) are partial generalizations of results by Chojnacki on the cosine equation.

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1. Introduction

Let $(G, +)$ be an abelian group with neutral element 0. The *cosine equation*, also called d'Alembert's equation, on G is the equation

$$\frac{f(x+y)+f(x-y)}{2} = f(x)f(y), \quad x, y \in G, \tag{1.1}$$

where $f : G \to \mathbb{C}$ is the unknown.

The present paper deals with an extension of (1.1), both with respect to the form of (1.1) where a transformation group will enter, and to the range of f which will be in the bounded operators on a Hilbert space. The non-zero solutions of (1.1) are the functions of the form $f(x) = (\gamma(x)+\gamma(-x))/2$, $x \in G$, where γ is a homomorphism of G into the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers [19, Theorem 2]. If the solution is bounded, then γ is a homomorphism of G into the multiplicative group $\mathbb{T}$ of complex numbers of modulus 1. We shall generalize this last fact.

Let G be an abelian topological group and let K be a compact topological transformation group of G, acting by automorphisms of G. Writing the action by $k \in K$ on $x \in G$ as $k \cdot x$ and letting dk denote the normalized Haar measure on K a generalization of the cosine equation (1.1) is

$$\int_K f(x + k \cdot y)\, dk = f(x)f(y), \quad x, y \in G, \tag{1.2}$$

where $f \in C(G)$ is the unknown. Eq. (1.2) is studied in the theory of group representations, being the relation defining K-spherical functions (for the terminology see [3, p. 88]).

Here are some examples: The Cauchy equation $f(x+y) = f(x)f(y)$ has $K = \{I\}$. The cosine equation (1.1) is the case of $K = \mathbb{Z}_2 = \{\pm 1\}$, where the action by $-1 \in \mathbb{Z}_2$ on G is the group inversion. However, in our set up this action can be any involutive automorphism of $\sigma : G \to G$. The cosine equation then becomes

$$\frac{f(x+y)+f(x+\sigma(y))}{2} = f(x)f(y), \quad x, y \in G. \tag{1.3}$$

The non-zero solutions of (1.3) are the functions of the form $f = (\gamma + \gamma \circ \sigma)/2$, where γ is a homomorphism of G into $\mathbb{C}^*$ [2, Theorem 3, 29].

Another example of K is $\mathbb{Z}_N = \{\omega^n \mid n = 0, \ldots, N-1\}$, where $\omega = \exp(2\pi i/N)$, acting on $\mathbb{R}^2 = \mathbb{C}$ by multiplication. A further one is $O(n)$ acting on $\mathbb{R}^n$ by rotations.

The first part of the following general Theorem 1.1 is due to Shin'ya [28, Corollary 3.12], and the second part to Chojnacki [6, Theorem 1.1].

Theorem 1.1. *Let G be a locally compact abelian Hausdorff topological group and let K be a compact topological transformation group of G, acting by automorphisms of G.*