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Divisors of modular forms on $\Gamma_0(4)$

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Abstract

There is a relationship between the values of a sequence of modular functions at points in the divisor of a meromorphic modular form and the exponents of its infinite product expansion. We make this relationship explicit for the case of modular forms on the congruence subgroup $\Gamma_0(4)$. We also consider some applications to classical number theoretic functions such as the number of representations of an integer as a sum of squares.

Keywords: Modular form; Modular function; Divisor; Theta operator; Genus zero congruence subgroups

1. Introduction and statement of results

For $f(z)$ meromorphic on the upper half-plane $\mathbb{H}$, define the theta operator by

$$\theta f(z) := \frac{1}{2\pi i} \frac{d}{dz} f(z). \tag{1.1}$$

Define $\sigma(n) := \sum_{d|n} d$, and denote by $E_2(z)$ the weight two Eisenstein series

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \qquad (q := e^{2\pi i z}). \tag{1.2}$$

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If f is a meromorphic modular form of weight $k \in \mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$, then it is known that

$$\frac{\theta f}{f} = \frac{k}{12} E_2 - f_\theta,$$

where f_θ is a meromorphic modular form of weight two. Bruinier et al. [2] determined f_θ explicitly in terms of the values of a sequence of modular functions. This gives interesting results regarding the exponents of product expansions of modular forms, and recurrence relations for their Fourier coefficients. Choie and Kohnen [3] examined some corresponding problems for functions modular on Hecke groups, and Ahlgren [1] considered the genus zero congruence subgroups $\Gamma_0(p)$ for $p \in \{2, 3, 5, 7, 13\}$.

After this series of papers (the general outline for studying such questions was provided by Brunier et al. [2]) it is natural to consider the analogous problem for $\Gamma_0(4)$. This case is of interest because it is the first where half-integral weight forms arise. This approach will clearly extend to other genus zero subgroups. We consider some applications to natural number theoretic functions; for example, representations of integers as sums of squares.

The space of all meromorphic (resp. holomorphic) modular forms of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(4)$ (see Section 2 for definition) will be denoted $\mathcal{M}_k$ (resp. M_k). Define $\Theta(z) \in M_{1/2}$ by

$$\Theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} \tag{1.3}$$

and $F_1(z) \in M_2$ by

$$F_1(z) := \sum_{\text{odd } n>0} \sigma(n) q^n.$$

Then, let

$$\phi(z) := \frac{\Theta^4(z)}{F_1(z)} = q^{-1} + 8 + 20q - 62q^3 + \cdots \ . \tag{1.4}$$

The function $\phi(z) \in \mathcal{M}_0$ is univalent with a simple pole at infinity and a simple zero at 1/2. For each positive integer m, define j_m as the unique modular function on $\Gamma_0(4)$, holomorphic on $\mathbb{H}$, vanishing at 1/2, and whose Fourier expansion at infinity has the form

$$q^{-m} + c_m(0) + c_m(1)q + \text{higher-order terms.} \tag{1.5}$$