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Preservation of defect sub-schemes by the action of the tangent sheaf

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Abstract

Let X/S be a noetherian scheme with a coherent $\mathcal{O}_X$-module M, and $T_{X/S}$ be the relative tangent sheaf acting on M. We give constructive proofs that sub-schemes Y, with defining ideal I_Y, of points $x \in X$ where $\mathcal{O}_x$ or M_x is “bad”, are preserved by $T_{X/S}$, making certain assumptions on X/S. Here bad means one of the following: $\mathcal{O}_x$ is not normal; $\mathcal{O}_x$ has high regularity defect; $\mathcal{O}_x$ does not satisfy Serre’s condition (R_n); $\mathcal{O}_x$ has high complete intersection defect; $\mathcal{O}_x$ is not Gorenstein; $\mathcal{O}_x$ does not satisfy (T_n); $\mathcal{O}_x$ does not satisfy (G_n); $\mathcal{O}_x$ is not n-Gorenstein; M_x is not free; M_x has high Cohen–Macaulay defect; M_x does not satisfy Serre’s condition (S_n); M_x has high type. Kodaira–Spencer kernels for syzygies are described, and we give a general form of the assertion that M is locally free in certain cases if it can be acted upon by $T_{X/S}$.
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1. Introduction

The tangent sheaf referred to in the title is the sheaf of derivations $T_{X/S} = Hom_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)$ of a noetherian scheme X/S, where $\Omega_{X/S}$ is the sheaf of relative differentials. Say that a point $x \in X$ is preserved by a derivation $\partial_x \in T_{X/S,x}$ if $\partial_x(\mathfrak{m}_x) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_x$. Say that a local section ∂ of $T_{X/S}$ preserves

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a subset W of X if its germ ∂_x preserves x when x is minimal prime in W, and if V is a sub-scheme of X with defining ideal I_V, then ∂ preserves V if ∂_x preserves the stalk $I_{V,x}$ when x is an associated prime of V; this is equivalent to having $\partial(I_V) \subset I_V$.

Let $\mathbf{P}(x)$ be a property of a point x in X and put

$$\mathbf{P}(X) = \{x \in X \mid \mathbf{P}(x)\}, \quad \text{and}$$
$$\mathbf{SP}(X) = X \backslash \mathbf{P}(X).$$

A property $\mathbf{P}$ is "good" for X if $\mathbf{P}(X)$ is open and dense, so one can regard $\mathbf{SP}(X)$ as a defect set, or a "singular" subset of X.

The following examples of defect sets $\mathbf{SP}_i(X)$ will be considered ($n = 0, 1, 2, \ldots$):

($\mathbf{P}_1$) $\{x \in X \mid \mathcal{O}_x$ is normal$\}$;
($\mathbf{P}_2$) (Regularity defect) $\{x \in X \mid \text{emb dim } \mathcal{O}_x - \dim \mathcal{O}_x \leqslant n\}$;
($\mathbf{P}_3$) (Serre's condition (R_n)) $\{x \in X \mid \mathcal{O}_x$ satisfies$(R_n)\}$;
($\mathbf{P}_4$) (Complete intersection defect) $\{\varepsilon_1(\mathcal{O}_x) - (\text{emb dim } \mathcal{O}_x - \dim \mathcal{O}_x) \leqslant n\}$, where $\varepsilon_1(\mathcal{O}_x) = \dim_k H_1(K^\bullet)$ and $K^\bullet$ is the Koszul complex of a minimal basis of the maximal ideal $\mathfrak{m}_x$;
($\mathbf{P}_5$) (Cohen–Macaulay defect) $\{x \in X \mid \dim M_x - \text{depth } M_x \leqslant n\}$, where M is a coherent $\mathcal{O}_X$-module;
($\mathbf{P}_6$) (Serre's condition (S_n)) $\{x \in X \mid M_x$ satisfies $(S_n)\}$;
($\mathbf{P}_7$) (Type) $\{x \in X \mid \dim_{k_x} Ext^t_{\mathcal{O}_x}(k_x, M_x) < n\}$, where $t = \text{depth } M_x$;
($\mathbf{P}_8$) $\{x \in X \mid \mathcal{O}_x$ is Gorenstein$\}$;
($\mathbf{P}_9$) $\{x \in X \mid \mathcal{O}_x$ satisfies$(T_n)\}$;
($\mathbf{P}_{10}$) $\{x \in X \mid \mathcal{O}_x$ satisfies$(G_n)\}$;
($\mathbf{P}_{11}$) $\{x \in X \mid \mathcal{O}_x$ is n-Gorenstein$\}$.

It is of course plausible that the defect sets $\mathbf{SP}(X)$ are preserved by $T_{X/S}$, thinking of sections of $T_{X/S}$ as infinitesimal automorphisms, and there exist several positive results. A. Seidenberg obtained that $\mathbf{SP}_1(X)$ is preserved, using Hasse–Schmidt differentiations [20], and that $\mathbf{SP}_2(X)$ is preserved when $n = 0$ and X is of finite type over a field, using "Zariski's lemma" [19]. Rational numbers are needed in both cases, since then any derivation comes from a differentiation, but Seidenberg noted that $\mathbf{SP}_1(X)$ is always preserved by differentiations, without the need for rational numbers. This idea of using differentiations instead of derivations was extended by Matsumura [15, Theorem 4] (see also [16, Theorem 32.2]) to get, in the case $n = 0$, that $\mathbf{SP}_2(X)$, $\mathbf{SP}_4(X)$, $\mathbf{SP}_5(X)$, and $\mathbf{SP}_8(X)$ are preserved by differentiations in quite a general situation, in any characteristic; a remaining difficulty is of course the determination of which derivations are integrable to differentiations when rational numbers are not available [15]. Assuming that $\mathcal{O}_S$ contains the rational numbers, the preservation of $\mathbf{SP}_2(X)$, $\mathbf{SP}_4(X)$, $\mathbf{SP}_5(X)$, $\mathbf{SP}_7(X)$ (for any n) was obtained by Kunz [12] in an equally general form as Seidenberg and Matsumura, extending an idea by Hart [11] (profitting from [19,20]), who proved that $\mathbf{SP}_2(X)$ is preserved in the absolute case. Using cases (5) and (7) we also get the preservation of $\mathbf{SP}_8(X)$ (Theorem 3.6.1), without (explicitly) using differentiations. However, the method of Seidenberg, Hart and Kunz, using Zariski's lemma, and Seidenberg and Matsumura, using differentiations, do not give the whole picture. As already explained, rational numbers are essential in the cited work,

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