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Journal of Pure and Applied Algebra 201 (2005) 17–24

JOURNAL OF PURE AND APPLIED ALGEBRA

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A note on the Rees algebra of a bipartite graph☆

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Received 1 March 2002; received in revised form 2 July 2002
Available online 1 February 2005

Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be an ideal of R generated by a set $x^{\alpha_1}, \ldots, x^{\alpha_q}$ of square-free monomials of degree two such that the graph G defined by those monomials is bipartite. We study the Rees algebra $\mathscr{R}(I)$ of I, by studying both the Rees cone $\mathbb{R}_+\mathscr{A}'$ generated by the set $\mathscr{A}' = \{e_1, \ldots, e_n, (\alpha_1, 1), \ldots, (\alpha_q, 1)\}$ and the matrix C whose columns are the vectors in $\mathscr{A}'$. It is shown that C is totally unimodular. We determine the irreducible representation of the Rees cone in terms of the minimal vertex covers of G. Then we compute the a-invariant of $\mathscr{R}(I)$.

MSC: Primary: 13H10; secondary: 13F20; 13A30; 52B20

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let G be a bipartite graph with vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ and edge set $E = E(G)$. The *edge ideal* of G is the square-free monomial ideal of R given by

$$I = I(G) = (\{x_i x_j | \{v_i, v_j\} \text{ is an edge of } G\}) \subset R,$$

☆ The authors were partially supported by CONACyT Grant 40201F and SNI.

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.12.013

and the *Rees algebra* of I is the K-subalgebra:

$$\mathscr{R}(I) = K[\{x_i x_j t \mid v_i \text{ is adjacent to } v_j\} \cup \{x_1, \ldots, x_n\}] \subset R[t],$$

where t is a new variable. Consider the set of vectors

$$\mathscr{A}' = \{e_i + e_j + e_{n+1} \mid v_i \text{ is adjacent to } v_j\} \cup \{e_1, \ldots, e_n\} \subset \mathbb{R}^{n+1},$$

where e_i is the ith unit vector. Here we study $\mathscr{R}(I)$ by looking closely at the matrix C whose columns are the vectors in $\mathscr{A}'$. One of our results proves that C is totally unimodular. Then as a consequence we derive that the presentation ideal of $\mathscr{R}(I)$ is generated by square-free binomials. As another consequence we give a simple proof of the fact that $\mathscr{R}(I)$ is a normal domain [7].

We are able to determine the irreducible representation of the polyhedral *Rees cone* $\mathbb{R}_+\mathscr{A}'$ generated by $\mathscr{A}'$, see Corollary 4.3. This turns out to be related to the minimal vertex covers of the graph G and yields a description of the canonical module of $\mathscr{R}(I)$.

By assigning $\deg(x_i) = 1$ and $\deg(t) = -1$, the Rees algebra $\mathscr{R}(I)$ becomes a standard graded K-algebra, i.e., it is generated as a K-algebra by elements of degree 1. Another of our results proves that the a-invariant of $\mathscr{R}(I)$, with respect to this grading, is equal to $-(\beta_0 + 1)$, where β_0 is the independence number of G. In order to compute this invariant we use the irreducible representation of $\mathbb{R}_+\mathscr{A}'$ together with a formula of Danilov–Stanley for the canonical module of $\mathscr{R}(I)$.

2. Preliminaries

Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials of R and let $A = (a_{ij})$ be the matrix of order $n \times q$ whose columns are the vectors $\alpha_1, \ldots, \alpha_q$. We say that the matrix A is *unimodular* if all its nonzero $r \times r$ minors have absolute value equal to 1, where r is the rank of A.

Recall that the monomial subring $K[F] \subset R$ is *normal* if $K[F] = \overline{K[F]}$, where $\overline{K[F]}$ is the integral closure of $K[F]$ in its field of fractions. The following expression for the integral closure is well known:

$$\overline{K[F]} = K[\{x^a \mid a \in \mathbb{Z}\mathscr{A} \cap \mathbb{R}_+\mathscr{A}\}], \tag{1}$$

where $\mathbb{Z}\mathscr{A}$ is the subgroup spanned by $\mathscr{A}$ and $\mathbb{R}_+\mathscr{A}$ is the *polyhedral cone* generated by $\mathscr{A} = \{\alpha_1, \ldots, \alpha_q\}$:

$$\mathbb{R}_+\mathscr{A} = \left\{ \sum_{i=1}^{q} a_i \alpha_i \,\middle|\, a_i \in \mathbb{R}_+ \quad \text{for all } i \right\}.$$

Here $\mathbb{R}_+$ denotes the set of nonnegative real numbers. See [2,3] and [12, Chapter 7] for a thorough discussion of the integral closure of a monomial subring and how it can be computed. For the related problem of computing the integral closure of an affine domain see [9,10].

The next result was shown in [7] if A is the incidence matrix of a bipartite graph and it was shown in [8] for general A. The proof below, in contrast to that of [8], is direct and does not make any use of Gröbner bases techniques.