Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Pure and Applied Algebra 195 (2005) 281–292
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Gap sequences on Klein surfaces[☆]

Angel L. Perez del Pozo

Departamento de Álgebra, Facultad de Matemáticasm, Universidad Complutense de Madrid, Ciudad Universitaria s/n, 28040 Madrid, Spain

Received 30 January 2004; received in revised form 10 May 2004
Communicated by M.-F. Roy
Available online 25 September 2004

Abstract

In this work we provide a possible definition for the gap sequence at a point of a compact Klein surface in an attempt to generalize the notion of Weierstrass gap sequence at a point of a compact Riemann surface. We obtain some results about the properties of these gap sequences and use them to study the G_n sets consisting of the points which have n as its first non-gap. We prove that these sets are invariant under the action of the automorphisms of the surface. We show that there are Klein surfaces of arbitrary genus such that the set G_1 is non-empty (if this is the case, it is a semialgebraic subset of real dimension one). If a surface has this property, then it must be hyperelliptic. In this case, we find that the topology of the sets G_n determine the topological type of the surface.

MSC: 14H55; 30F50

1. Introduction

In this work we study Weierstrass points on compact symmetric Riemann surfaces and the Klein surfaces obtained as a quotient of them. The set of Weierstrass points is an interesting object of study because of its good properties; for example, every automorphism of X maps injectively the set of Weierstrass points onto itself.

We recall some basic facts about Weierstrass points in Section 2; most of them appear in [5]. In Section 3 we introduce the double covering of a Klein surface and the affine plane

☆ Partially supported by the European Network RAAG HPRN-CT-2001-00271 and the Spanish GAAR DGICYT BFM2002-04797.

E-mail address: angel_perez@mat.ucm.es (A.L. Perez del Pozo).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.08.002

model of a hyperelliptic Klein surface, which will be very useful later (more information about Klein surfaces can be found, for example, in [1,3]).

In Section 4 we study the gap sequence at a point of a Klein surface, and in Section 5 we use our previous results to distinguish some marked sets in a hyperelliptic Klein surface. The topological characteristics of these sets allow us to determine the topological type of the surface. We also prove a result which gives us information about the action of an automorphism on the set of Weierstrass points on the boundary of the surface.

2. Weierstrass points

We fix all throughout this section a compact Riemann surface X of genus $g \geqslant 0$. Let $\mathscr{M}(X)$ denote the field of meromorphic functions on X.

Definition 2.1. Let $P \in X$ and $n \in \mathbb{N}$. A meromorphic function on X with a pole of order n at P and no other poles in X will be called (following the notation used by Lewittes in [7]) a *function at* $n(P)$.

The Weierstrass gap Theorem describes, for a point $P \in X$, the nature of the set of positive integers n such that there exists a function at $n(P)$. In fact:

Theorem 2.2 (*Weierstrass gap Theorem*). *Let X be a compact Riemann surface of genus $g \geqslant 1$ and $P \in X$. Then, there are precisely g integers $1 = \gamma_1 < \gamma_2 < \cdots < \gamma_g < 2g$ such that there exists a function at $n(P)$ if and only if $n \notin \Gamma(P) = \{\gamma_1, \ldots, \gamma_g\}$.*

The set $\Gamma(P) = \{\gamma_1, \ldots, \gamma_g\}$ is called the *gap sequence* at P. The integers γ_i are called the *gaps* at P. An integer in $\mathbb{N} \backslash \Gamma(P)$ is called a *non-gap* at P. A point P is a *Weierstrass point* if $\Gamma(P) \neq \{1, 2, \ldots, g\}$.

If X is a compact Riemann surface of genus $g \geqslant 2$, then the set W of Weierstrass points on X is finite. Moreover, $2g + 2 \leqslant |W| \leqslant g^3 - g$. The lower bound is attained if and only if X is hyperelliptic.

The Weierstrass gap Theorem can be obtained as a corollary of the Noether gap Theorem, which we will need later. To introduce it, we have to use the language of divisors. A *divisor* on X is a formal symbol

$$D = \sum_{P \in X} D_P \cdot P$$

with $D_P \in \mathbb{Z}$ and $D_P \neq 0$ for only finitely many $P \in X$. Divisors on X form an abelian group under pointwise addition. In fact, they constitute the free abelian group on the set of points of X.

For a divisor D on X, the $\mathbb{C}$-*vector space of meromorphic functions with poles bounded by* D, which we will denote by $L_X(D)$ (or simply by $L(D)$ if X is understood), is the set of