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Degree formulae for offset curves☆

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Abstract

In this paper, we present three different formulae for computing the degree of the offset of a real irreducible affine plane curve $\mathscr{C}$ given implicitly, and we see how these formulae particularize to the case of rational curves. The first formula is based on an auxiliary curve, called $\mathscr{S}$, that is defined depending on a non-empty Zariski open subset of $\mathbb{R}^2$. The second formula is based on the resultant of the defining polynomial of $\mathscr{C}$, and the polynomial defining generically $\mathscr{S}$. The third formula expresses the offset degree by means of the degree of $\mathscr{C}$ and the multiplicity of intersection of $\mathscr{C}$ and the hodograph $\mathscr{H}$ to $\mathscr{C}$, at their intersection points.

MSC: 14Q10; 68W30; 14E05

1. Introduction

Some applications in computer-aided geometric design (CAGD) require the manipulation of certain geometric objects called offsets (see [13,5,10]). These objects are algebraic varieties, in fact hypersurfaces, that essentially appear when taking the envelope of a system of hyperspheres with fixed, but probably undetermined, distance and centered at the points of a given hypersurface.

The formal definition of offset can be introduced as follows. We focus here on the notion of offset to a complex affine plane curve, for the concept of offset to a hypersurface over

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an arbitrary algebraically closed field, of characteristic zero, we refer to [22]. We consider $\mathbb{C}^2$ as the metric affine space, induced by the inner product $B(X, Y) = X \cdot I \cdot Y^{\mathrm{T}}$ defined by 2×2 identity matrix I; note that the metric we are using is not the hermitic standard one. Now, let $\mathscr{C}$ be an irreducible affine plane curve over $\mathbb{C}$, and let $\mathscr{C}_0 \subset \mathscr{C}$ be the set of regular points p of $\mathscr{C}$ such that any non-zero normal vector to $\mathscr{C}$ at p is non-isotropic; i.e. the norm, in the metric affine space $\mathbb{C}^2$, of the normal vector is non-zero. Then, the *offset* to $\mathscr{C}$, at distance d, is the Zariski closure of the constructible set $\mathscr{A}_d(\mathscr{C})$ consisting of the intersection points of the circles of radius d centered at each point $p \in \mathscr{C}_0$ and the normal line to $\mathscr{C}$ at p. We denote the offset to $\mathscr{C}$ at distance d as $\mathscr{O}_d(\mathscr{C})$. We observe that, if $\mathscr{C}$ is given by a rational parametrization $\mathscr{P}(t)$, then $\mathscr{A}_d(\mathscr{C})$ is essentially the set in $\mathbb{C}^2$ generated by the formula $\mathscr{P}(t) \pm d\frac{\mathscr{N}(t)}{\|\mathscr{N}(t)\|}$, where $\mathscr{N}(t)$ is the normal vector to $\mathscr{C}$ associated with the parametrization $\mathscr{P}(t)$. In this expression, by abuse of notation, for every non-isotropic $X \in \mathbb{C}^2$ we write $\|X\|$ to express any of the two numbers such that $\|X\|^2 = B(X, X)$; if $X \in \mathbb{C}^2$ is isotropic, then we write $\|X\| = 0$.

Note that, if $\mathscr{C}_0$ does not contain infinitely many points, then the offsetting construction yields the empty set. However, the only irreducible plane curves over $\mathbb{C}$ with this property are the lines passing through the cyclic points (see [20]); in particular if $\mathscr{C}$ is a real curve, this phenomenon does not occur. In addition, it holds that, if $\mathscr{C}$ is not one of these lines, then $\mathscr{O}_d(\mathscr{C})$ has dimension 1 (see [22]). Moreover, if $\mathscr{C}$ is not a circle or $\mathscr{C}$ is a circle and d is not the radius, then $\mathscr{O}_d(\mathscr{C})$ is an affine plane curve with at most two components (see [22]).

Although offset curves were already introduced by Leibniz in [14], under the term of parallel curve, it is only from the 1980s when, as a consequence of the development of CAGD, the study of offsets to hypersurfaces turns to be an active research area. Indeed, as a consequence of this research, many interesting questions directly related to algebraic geometry have appeared. In particular, implicitization problems have been considered in [11,12,24], and parametrization problems have been addressed in [1,16–18,21]). Furthermore, algebraic, and geometric properties of the offsets in terms of the corresponding properties of the initial variety have been analyzed; for instance singularities, self-intersections are studied in [7,8], a formula for the genus of the offset in terms of the degree and genus of the original curve have been presented in [2], and the degeneration analysis of the offsetting construction can be found in [22].

However, topological questions, and the problem of relating the degree of the offset to the degree of the original variety have not been studied so extensively. In this paper we deal with the second problem, that is, with the problem of giving formulae that provide explicitly the degree of $\mathscr{O}_d(\mathscr{C})$ in terms of the degree of $\mathscr{C}$. Let us mention that the question on the degree plays an important role in some applications as the computation of the Voronoi diagram of obstacles whose borders are curved (see [4]). The offset degree problem was already studied by classical geometers (see, for instance, [15, pp. 643–651]) by means of the Plücker formulae, and some results for the case of conics and for special families of curves (such as epicycloids and hypocycloids) were derived. More recently, in [8] the authors give a degree formula for the case of rational real curves based on gcds of univariate polynomials. However, no formulae have been presented for the case of curves given implicitly.

In this paper, we treat the general case and we provide offset degree formulae for algebraic real curves non-necessarily rational and therefore given implicitly. More precisely,