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On products and duality of binary, quadratic, regular operads

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Abstract

Since its introduction by Loday in 1995, with motivation from algebraic K-theory, dendriform dialgebras have been studied quite extensively with connections to several areas in Mathematics and Physics. A few more similar structures have been found recently, such as the tri-, quadri-, ennea- and octo-algebras, with increasing complexity in their constructions and properties. We consider these constructions as operads and their products and duals, in terms of generators and relations, with the goal to clarify and simplify the process of obtaining new algebra structures from known structures and from linear operators.

MSC: 18D50; 16W30

1. Introduction

In order to study the periodicity of algebraic K-groups, J.-L. Loday laid out a program in [30] which led him to the concepts of associative dialgebra and dendriform dialgebra [31]. In the next few years, their properties were studied by several authors in areas related to operads [35], homology [17,18], Hopf algebras [7,26,42,39], combinatorics [16,37,2,3], arithmetic [34] and quantum field theory [16]. See [33] and other articles in the volume for a survey of some of these developments.

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Since 2002, quite a few more similar algebra structures have been introduced, such as the associative trialgebra and dendriform trialgebra of Loday and Ronco [38], the dendriform quadri-algebra of Aguiar and Loday [4], the ennea-algebra, the NS-algebra, the dendriform-Nijenhuis algebra and the octo-algebra of Leroux [27–29]. These algebras have a common property of "splitting associativity", i.e., expressing the multiplication of an associative algebra as the sum of a string of binary operations. The operations in the string satisfy a set of relations and the associativity of the multiplication follows from the sum of these relations. The first instance of such algebras, the dendriform dialgebra, has a string of two operators. The later constructions were largely inspired by the connection [1,10,27] with Rota–Baxter operators[1] which were introduced by Baxter [6] in 1960 and were actively studied in the 1970s [43,44] and again in recent years in connection with several areas of Mathematics and Physics [5,8,9,11,12,14,15,21–25].

Two themes can be found in these recent constructions. One is the construction of a new type of algebra that has the combined features of types of two or more algebras that were previously known. The other is the use of a linear operator with certain features, such as a Rota–Baxter operator, on a known type of algebras to obtain another type of algebra with richer structures. Even though the ideas of the themes are simple, to carry them out for a particular construction can be quite complicated.

The purpose of this paper is to study these constructions in the framework of operads and their products, given by generators and relations. This enables us to clarify, simplify and further generalize the constructions and properties of these recent algebra structures.

Here is a more detailed plan of the paper. In Section 2, we recall the operads that give rise to the above algebra structures. These operads are binary, quadratic and regular [35] operads with a splitting of associativity. To ease the notation, we call them ABQR operads and the corresponding algebras ABQR algebras. The generator-relation construction of ABQR operads allows a concrete description which is quite simple and can be found in the existing literature [33]. We use this description to formally define the types of such operads.

We then define in Section 3 products of ABQR algebras that are similar to (but different from) the operad products of Manin–Ginzburg–Kapranov [40,41,20,32]. We show, formalizing the first theme, that some recently obtained ABQR algebras [4,27–29] are products of simpler algebras. Properties of products of ABQR algebras are also studied in this section. The subsequent Section 4 considers the dual of a ABQR operad and its relation with the products.

A full understanding of the second theme mentioned above depends on sufficient knowledge of the linear operators that give rise to ABQR algebras. While this topic is being investigated in another project, using the framework introduced here, we can make the second theme precise for most of the operators that we are aware of, when the operators are applied to any type of ABQR algebras. This is presented in Section 5.

The concept of unit actions of operads has recently been introduced by Loday [35] and used to construct Hopf algebras on the free algebras. In a separate work [13], we investigate the relation between products of operads and their unit actions. See [45] also for a more recent application of products of ABQR operads.

[1] They used to be called Baxter operators. They are renamed Rota–Baxter operators to distinguish it clearly from the very related Yang–Baxter operators. The latter Baxter is the Australian physicist Rodney Baxter.