Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 402 (2005) 228–244
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Recognizing linear structure in noisy matrices ☆

Marianna Bolla *

Institute of Mathematics, Budapest University of Technology and Economics, P.O. Box 91, Bldg. H.5/7, 1521 Budapest, Hungary

Received 4 November 2004; accepted 29 December 2004
Available online 3 February 2005
Submitted by R.A. Brualdi

Abstract

Behaviour of the eigenvalues of random matrices with an underlying linear structure is investigated, when the structure is exposed to random noise. The question, how a deterministic skeleton behind a random matrix can be recognized, is also discussed. Such random matrices, as weight matrices of random graphs, adequately describe some large biological and communication networks. A range for the power of random power law graphs—for which the structure is robust enough—is established.

AMS classification: 15A52; 60F10

Keywords: Wigner-noise; Blown up matrices; Perturbation of the eigenvalues; Large deviations

1. Introduction

Mostly we think of random matrices as completely random Wigner-type matrices whose eigenvalues obey the semi-circle law. No matter how important this type of a matrix in quantum mechanics was, in case of real-life matrices it is merely a random noise added to the underlying linear structure of the matrix (if there is any).

☆ Research supported by the Foundation for Research Development in Hungary (NKFP) Grant No. 2/0017/2002.
* Tel.: +36 1 200 0646; fax: +36 1 463 1677.
E-mail address: marib@math.bme.hu

0024-3795/$ - see front matter
doi:10.1016/j.laa.2004.12.023

Although, it is hard to recognize the structure concealed by the noise, in a number of models it is possible by means of spectral techniques and large deviations principles.

Usually, our matrix is the weight matrix of some random weighted graph $G = (V, \mathbf{A})$ with an n-element vertex set V and $n \times n$ symmetric weight matrix $\mathbf{A}$, where $n \to \infty$. For example, some communication, social or biological networks can be adequately described by a random graph model. Performing graph-embedding techniques, it is a crucial question how many protruding eigenvalues—with corresponding eigenvectors—to choose for the vertex-representation.

Also, the classical numerical algorithms for the spectral decomposition of a matrix with size exceeding a million are not immediately applicable, and some newly developed randomized algorithms are to be used instead, see [1]. These algorithms exploit the randomness of our matrix, and rely on the fact that a random noise will not change the order of magnitude of the relevant eigenvalues with large absolute value. Sometimes—instead of depriving our matrix of the noise—a noise is added (by digitalizing the entries of or making the underlying matrix sparse by an appropriate randomization) to make the matrix more easily decomposable by means of the classical methods. For example, the Lánczos method (see Section 9 of [11]) is applicable to large, sparse, symmetric eigenproblems.

Both the number of eigenvalues to be kept and algorithmic questions can be—at least partly—analyzed by means of the results in Sections 2 and 3. For an easy discussion, in [6] we introduced the notion of Wigner-noise that is a generalization of a random matrix investigated by Wigner [15] and the eigenvalues of which obey the semi-circle law (if the order of the matrix tends to infinity). We cite the definitions.

Definition 1.1. The $n \times n$ real matrix $\mathbf{W}$ is a *Wigner-noise* if it is symmetric, its entries w_{ij}, $1 \leqslant i \leqslant j \leqslant n$, are independent random variables, $\mathbb{E}(w_{ij}) = 0$, $\mathrm{Var}(w_{ij}) \leqslant \sigma^2$ with some $0 < \sigma < \infty$ and either the w_{ij}'s are uniformly bounded (there is a constant $K > 0$ such that $|w_{ij}| \leqslant K$) or they are Gaussian distributed.

For example, mutations in cellular networks as well as random effects in social networks can be modelled by a Wigner-noise. By the method of Füredi and Komlós [10] it can be proved (see [1]) that for the maximum absolute value eigenvalue of $\mathbf{W}$

$$\max_{1 \leqslant i \leqslant n} |\lambda_i(\mathbf{W})| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3} \log n) \tag{1.1}$$

holds with probability tending to 1, if $n \to \infty$.

In the sequel, we put this noise on the following general deterministic structure.

Definition 1.2. The $n \times n$ matrix $\mathbf{B}$ is a *blown up matrix*, if there is a constant $k < n$, a $k \times k$ symmetric so-called *pattern matrix* $\mathbf{P}$ with entries $0 \leqslant p_{ij} \leqslant 1$, and there are positive integers $n_1, \ldots, n_k$, $\sum_{i=1}^{k} n_i = n$ such that $\mathbf{B}$ can be divided into k^2 blocks, the block (i, j) being an $n_i \times n_j$ matrix with entries all equal to p_{ij} $(1 \leqslant i, j \leqslant k)$.