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Linear Algebra and its Applications 403 (2005) 118–142
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Tridiagonal pairs of height one

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Received 13 December 2004; accepted 21 January 2005
Available online 19 March 2005
Submitted by R.A. Brualdi

Abstract

Let (A, A^*) denote a tridiagonal pair on a vector space V over a field $\mathbb{K}$. Let $V_0, \ldots, V_d$ denote a standard ordering of the eigenspaces of A on V, and let $\theta_0, \ldots, \theta_d$ denote the corresponding eigenvalues of A. We assume $d \geqslant 3$. Let q denote a scalar taken from the algebraic closure of $\mathbb{K}$ such that $q^2 + q^{-2} + 1 = (\theta_3 - \theta_0)/(\theta_2 - \theta_1)$. We assume q is not a root of unity. Let ρ_i denote the dimension of V_i. The sequence $\rho_0, \rho_1, \ldots, \rho_d$ is called the *shape* of the tridiagonal pair. It is known there exists a unique integer h $(0 \leqslant h \leqslant d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leqslant i \leqslant h$, $\rho_{i-1} = \rho_i$ for $h < i \leqslant d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \leqslant d$. The integer h is known as the *height* of the tridiagonal pair. In this paper we show that the shape of a tridiagonal pair of height one with $\rho_0 = 1$ is either $1, 2, 2, \ldots, 2, 1$ or $1, 3, 3, 1$. In each case, we display a basis for V and give the action of A, A^* on this basis.

AMS classification: 05E30; 05E35; 33C45; 33D45

Keywords: Tridiagonal pair; Tridiagonal relation; Leonard pair

1. Introduction

The notion of a tridiagonal pair was introduced by Ito et al. [3], generalizing the notion of a Leonard pair which had been introduced by Terwilliger [6]. See

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.01.032

Terwilliger's lecture note [8] about Leonard pairs and tridiagonal pairs. A tridiagonal pair is defined as follows.

Definition 1.1 [3]. Let V denote a nonzero finite dimensional vector space over a field $\mathbb{K}$. By a *tridiagonal pair* on V, we mean a pair (A, A^*), where $A : V \longrightarrow V$ and $A^* : V \longrightarrow V$ are linear transformations that satisfy the following conditions.

(i) A and A^* are both diagonalizable on V.
(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

Remark 1.2. With reference to Definition 1.1, it is known that $d = \delta$ [3, Corollary 5.7]. The common number d of the eigenspaces is called the *diameter* of the tridiagonal pair.

Throughout this paper, we fix the following notation. Let $\mathbb{K}$ denote a field and let V denote a nonzero finite dimensional vector space over $\mathbb{K}$. Let (A, A^*) denote a tridiagonal pair on V with diameter $d \geqslant 3$. Let $V_0, V_1, \ldots, V_d$ (respectively $V_0^*, V_1^*, \ldots, V_d^*$) denote an ordering of the eigenspaces of A (respectively A^*) that satisfies the condition (ii) (respectively (iii)) in Definition 1.1. Let ρ_i denote the dimension of V_i. Let θ_i (respectively θ_i^*) denote the eigenvalue of A (respectively A^*) for the eigenspace V_i (respectively V_i^*). Set $\beta = (\theta_3 - \theta_0)/(\theta_2 - \theta_1) - 1$, and let q denote a scalar taken from the algebraic closure $\overline{\mathbb{K}}$ such that $\beta = q^2 + q^{-2}$. We assume q is not a root of unity.

It is known [5, Theorem 3.3] there exists a unique integer h $(0 \leqslant h \leqslant d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leqslant i \leqslant h$, $\rho_{i-1} = \rho_i$ for $h < i \leqslant d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \leqslant d$. The integer h is known as the *height* of the tridiagonal pair.

Our first main result is the following.

Theorem 1.3. *Suppose $h = 1$ and $\rho_0 = 1$. Then one of the following holds.*

(i) $\rho_0 = 1,\ \rho_1 = \rho_2 = \cdots = \rho_{d-1} = 2,\ \rho_d = 1,$
(ii) $d = 3,\ \rho_0 = 1,\ \rho_1 = \rho_2 = 3,\ \rho_3 = 1.$