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Tridiagonal pairs of height one

Kazumasa Nomura

College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa 272-0827, Japan

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Abstract

Let (A, A^*) denote a tridiagonal pair on a vector space *V* over a field K. Let V_0, \ldots, V_d denote a standard ordering of the eigenspaces of *A* on *V*, and let $\theta_0, \ldots, \theta_d$ denote the corresponding eigenvalues of *A*. We assume $d \ge 3$. Let *q* denote a scalar taken from the algebraic closure of K such that $q^2 + q^{-2} + 1 = (\theta_3 - \theta_0)/(\theta_2 - \theta_1)$. We assume *q* is not a root of unity. Let ρ_i denote the dimension of V_i . The sequence $\rho_0, \rho_1, \ldots, \rho_d$ is called the *shape* of the tridiagonal pair. It is known there exists a unique integer h ($0 \le h \le d/2$) such that $\rho_{i-1} < \rho_i$ for $1 \le i \le h$, $\rho_{i-1} = \rho_i$ for $h < i \le d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \le d$. The integer *h* is known as the *height* of the tridiagonal pair. In this paper we show that the shape of a tridiagonal pair of height one with $\rho_0 = 1$ is either 1, 2, 2, ..., 2, 1 or 1, 3, 3, 1. In each case, we display a basis for *V* and give the action of *A*, *A** on this basis.

AMS classification: 05E30; 05E35; 33C45; 33D45

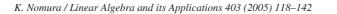
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1. Introduction

The notion of a tridiagonal pair was introduced by Ito et al. [3], generalizing the notion of a Leonard pair which had been introduced by Terwilliger [6]. See

E-mail address: nomura.las@tmd.ac.jp

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Terwilliger's lecture note [8] about Leonard pairs and tridiagonal pairs. A tridiagonal pair is defined as follows.

Definition 1.1 [3]. Let V denote a nonzero finite dimensional vector space over a field \mathbb{K} . By a *tridiagonal pair* on V, we mean a pair (A, A^*) , where $A : V \longrightarrow V$ and $A^* : V \longrightarrow V$ are linear transformations that satisfy the following conditions.

- (i) A and A^* are both diagonalizable on V.
- (ii) There exists an ordering V_0, V_1, \ldots, V_d of the eigenspaces of A such that

 $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \le i \le d),$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_{\delta}^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \le i \le \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than W = 0 and W = V.

Remark 1.2. With reference to Definition 1.1, it is known that $d = \delta$ [3, Corollary 5.7]. The common number *d* of the eigenspaces is called the *diameter* of the tridiagonal pair.

Throughout this paper, we fix the following notation. Let \mathbb{K} denote a field and let V denote a nonzero finite dimensional vector space over \mathbb{K} . Let (A, A^*) denote a tridiagonal pair on V with diameter $d \ge 3$. Let V_0, V_1, \ldots, V_d (respectively $V_0^*, V_1^*, \ldots, V_d^*$) denote an ordering of the eigenspaces of A (respectively A^*) that satisfies the condition (ii) (respectively (iii)) in Definition 1.1. Let ρ_i denote the dimension of V_i . Let θ_i (respectively θ_i^*) denote the eigenvalue of A (respectively A^*) for the eigenspace V_i (respectively V_i^*). Set $\beta = (\theta_3 - \theta_0)/(\theta_2 - \theta_1) - 1$, and let q denote a scalar taken from the algebraic closure $\overline{\mathbb{K}}$ such that $\beta = q^2 + q^{-2}$. We assume q is not a root of unity.

It is known [5, Theorem 3.3] there exists a unique integer h ($0 \le h \le d/2$) such that $\rho_{i-1} < \rho_i$ for $1 \le i \le h$, $\rho_{i-1} = \rho_i$ for $h < i \le d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \le d$. The integer h is known as the *height* of the tridiagonal pair.

Our first main result is the following.

Theorem 1.3. Suppose h = 1 and $\rho_0 = 1$. Then one of the following holds.

(i) $\rho_0 = 1$, $\rho_1 = \rho_2 = \dots = \rho_{d-1} = 2$, $\rho_d = 1$, (ii) d = 3, $\rho_0 = 1$, $\rho_1 = \rho_2 = 3$, $\rho_3 = 1$. 119

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