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Order preserving maps on the poset of upper triangular idempotent matrices

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Abstract

We obtain the general form of bijective order and orthogonality preserving maps on the poset of all $n \times n$ upper triangular idempotent matrices over an arbitrary field $\mathbb{F}$. We also show that the assumption of surjectivity can be removed if every nonzero homomorphism $g : \mathbb{F} \to \mathbb{F}$ is surjective.

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1. Introduction and statements of the main results

Let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on a Hilbert space H. Motivated by some problems in quantum mechanics Ovchinnikov [6] characterized automorphisms of the poset of all idempotents in $\mathscr{B}(H)$, $\dim H \geqslant 3$. Recall that an automorphism of the poset is a bijective map preserving order in both directions. For another recent application of this result in mathematical physics we refer to [4].

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In the finite-dimensional case beside the full matrix algebra there are other important subalgebras and corresponding posets of idempotents. In this paper we will be interested in the poset of upper triangular idempotent matrices. Let us start with the notation. Throughout this note $\mathbb{F}$ will denote a field and $M_n(\mathbb{F})$ the algebra of all $n \times n$ matrices over $\mathbb{F}$. For $A \in M_n(\mathbb{F})$ we denote by A^{t} the transpose of A. The elements of the standard basis of $M_n(\mathbb{F})$ will be denoted by E_{ij}, $1 \leqslant i, j \leqslant n$. By $PT_n(\mathbb{F})$ we denote the set of all upper triangular idempotent matrices in $M_n(\mathbb{F})$ and by $PT_n^k(\mathbb{F})$ the set of all upper triangular idempotent matrices of rank k in $M_n(\mathbb{F})$. The set $PT_n(\mathbb{F})$ is known to be a poset with $P \leqslant Q$ if $PQ = QP = P$ for $P, Q \in PT_n(\mathbb{F})$. A map $\phi : PT_n(\mathbb{F}) \to PT_n(\mathbb{F})$ is order preserving if for every pair $P, Q \in PT_n(\mathbb{F})$ the relation $P \leqslant Q$ implies $\phi(P) \leqslant \phi(Q)$. A map ϕ preserves orthogonality if $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$ for every pair of upper triangular idempotents P and Q satisfying $PQ = QP = 0$.

If T is any invertible upper triangular matrix, then clearly, the map $P \mapsto TPT^{-1}$ preserves order on the poset $PT_n(\mathbb{F})$. Let h be any automorphism of the underlying field $\mathbb{F}$. The map $[p_{ij}] \mapsto [h(p_{ij})]$, $[p_{ij}] \in PT_n(\mathbb{F})$, also preserves order on the poset $PT_n(\mathbb{F})$. And finally, the flip map, that is the transposition over the antidiagonal

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ 0 & p_{22} & p_{23} & \cdots & p_{2n} \\ 0 & 0 & p_{33} & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_{nn} \end{bmatrix} \mapsto$$

$$P^f = \begin{bmatrix} p_{nn} & p_{n-1,n} & p_{n-2,n} & \cdots & p_{1n} \\ 0 & p_{n-1,n-1} & p_{n-2,n-1} & \cdots & p_{1,n-1} \\ 0 & 0 & p_{n-2,n-2} & \cdots & p_{1,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_{11} \end{bmatrix}$$

is order preserving map on the poset $PT_n(\mathbb{F})$ as well. To see this note that $P^f = JP^{\mathrm{t}}J$. Here $J = E_{n1} + E_{n-1,2} + \cdots + E_{1n}$ is an involution having all antidiagonal entries equal to 1, while all off-antidiagonal elements of J are zero. In [3, Theorem 1.1] the author showed that these maps generate the group of all automorphisms of the poset $PT_n(\mathbb{F})$ that preserve orthogonality. Note that an automorphism of the poset $PT_n(\mathbb{F})$ is a bijective map preserving order in both directions, that is, $P \leqslant Q$ if and only if $\phi(P) \leqslant \phi(Q)$ for every pair $P, Q \in PT_n(\mathbb{F})$. Comparing this result with the finite-dimensional part of the Ovchinnikov's theorem we see that we have an additional assumption of preserving orthogonality. It turned out that the upper triangular case is more complicated than the full matrix algebra case and that the group of all automorphisms of the poset $PT_n(\mathbb{F})$ does not have a nice structure [3, Section 4]. The aim of this paper is to prove two theorems which generalize [3, Theorem 1.1]. The first natural question here is whether the result of the author