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A discrete systems approach to cardinal spline Hermite interpolation

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Abstract

A cardinal spline Hermite interpolation problem is posed by specifying values, and $m-1$ derivatives, $m \geqslant 1$, at uniformly spaced knots t_k; it may be solved by means of a generalized spline function $w(t)$ (a standard spline function when $m=1$), piecewise a polynomial of degree $n-1=2m+p-1,\ \ p \geqslant 0$, with $w^{(j)}(t)$ continuous across the knots for $j=0,1,2,\ldots,m+p-1$. The problem is studied here for $p>0$ in the context of an $(m+p)$-dimensional system of linear recursion equations satisfied by the values of the m-th through $m+p-1$-st derivatives of $w(t)$ at the knots, whose homogeneous term involves a $p \times p$ matrix $\mathbf{A}$. In the case $m=1$ we relate the characteristic polynomial of $\mathbf{A}$ and certain controllability notions to the standard B-spline and we proceed to show how systems-theoretic ideas can be used to generate systems of basis splines for higher values of m.

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1. A class of discrete linear systems

We consider problems of cardinal Hermite spline interpolation, also described as interpolation problems with “multiple knots”, on an interval $[t_0, t_K] = [t_0, t_0 + T]$, $T > 0$. Such generalized spline interpolation problems have been widely studied; we refer in particular to the work of Schoenberg [14], Micchelli [12] and de Boor and Schoenberg [3] as well as de Boor’s text [2]. Our purpose in this paper is to develop this subject matter in an alternate setting, that of modern linear systems theory, allowing results to be viewed in a different light. Although we are primarily concerned with cardinal splines much of the work in the first part of the paper extends without difficulty to the case of nonuniform knots.

We designate the knots as $t_k,\ k = 0, 1, 2, \ldots, K$, with $t_k - t_{k-1} = h = T/K$, $k = 1, 2, \ldots, K$, and consider functions $w(t)$ on $[t_0, t_K]$ which reduce to polynomials of degree $\leqslant n - 1$ on each $[t_{k-1}, t_k)$; thus are solutions there of $\frac{\mathrm{d}^n w}{\mathrm{d}t^n} = 0$. Setting $W_j(t) = w^{(j-1)}(t),\ j = 1, 2, \ldots, n$, we have the equivalent first order form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{M}\, W(t), \quad \mathbf{M}_{j,\ell} = \begin{cases} 1, & \ell = j + 1,\ j = 1, 2, \ldots, n - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We define

$$W_k^+ = W(t_k+) = W(t_k), \quad W_k^- = W(t_k-), \quad k = 0, 1, 2, \ldots, K - 1,$$
$$W_K^- = W(t_K);$$

then W_{k-1}^+ serves as the initial state for $W(t)$ on $[t_{k-1}, t_k)$. Thus, on that interval,

$$W(t) = \mathrm{e}^{\mathbf{M}(t-t_{k-1})} W_{k-1}^+; \quad W_k^- = \mathrm{e}^{\mathbf{M}h} W_{k-1}^+ \equiv \mathbf{E}(h) W_{k-1}^+. \tag{1.2}$$

In (1.2) $\mathbf{E}(h)$ is the upper triangular $n \times n$ matrix with j-row, ℓ-column, entry

$$\mathbf{e}(h)_\ell^j = \frac{h^{\ell-j}}{(\ell - j)!}, \quad 1 \leqslant j \leqslant \ell \leqslant n. \tag{1.3}$$

We suppose, for integral $p \geqslant 1$, that $n = 2m + p$ (the case $p = 0$ will be treated in a separate article). Then

$$\mathbf{E}(h) = \begin{pmatrix} \mathbf{E}_{00}(h) & \mathbf{E}_{01}(h) & \mathbf{E}_{02}(h) \\ \mathbf{O} & \mathbf{E}_{11}(h) & \mathbf{E}_{12}(h) \\ \mathbf{O} & \mathbf{O} & \mathbf{E}_{22}(h) \end{pmatrix}, \tag{1.4}$$

where $\mathbf{E}_{00}(h)$ is $m \times m$, $\mathbf{E}_{01}(h)$ is $m \times p$, $\mathbf{E}_{02}(h)$ is $m \times m, \ldots, \mathbf{E}_{11}(h)$ is $p \times p, \ldots,$ and $\mathbf{O}$ is used generically to indicate the zero matrix of appropriate dimension. In the sequel we will use the simplified notation $\mathbf{E}, \mathbf{E}_{00}, \mathbf{E}_{01}, \ldots,$ unless we need to refer specifically to the dependence of these matrices on the step length h.

In this article a *generalized spline* is a function $w(t)$, as described prior to (1.1), such that, W_k^j denoting the j-th component of W_k,

$$W_k^{j+} = W_k^{j-} \equiv W_k^j, \quad k = 1, 2, \ldots, K,\ j = 1, \ldots, n - m = m + p. \tag{1.5}$$