Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER Linear Algebra and its Applications 408 (2005) 221–229
www.elsevier.com/locate/laa

The inverse mean problem of geometric and contraharmonic means

Yongdo Lim

Department of Mathematics, Kyungpook National University, Taegu 702-701, Republic of Korea

Received 21 January 2005; accepted 9 June 2005
Available online 18 August 2005
Submitted by C.-K. Li

Abstract

In this paper we solve the inverse mean problem of contraharmonic and geometric means of positive definite matrices (proposed in [W.N. Anderson, M.E. Mays, T.D. Morley, G.E. Trapp, The contraharmonic mean of HSD matrices, SIAM J. Algebra Disc. Meth. 8 (1987) 674–682])

$$\begin{cases} A = X + Y - 2(X^{-1} + Y^{-1})^{-1}, \\ B = X\#Y, \end{cases}$$

by proving its equivalence to the well-known nonlinear matrix equation $X = T - BX^{-1}B$ where $T = \frac{1}{2}(A + A\#(A + 8BA^{-1}B))$ is the unique positive definite solution of $X = A + 2BX^{-1}B$. The inverse mean problem is solvable if and only if $B \leqslant A$.

Keywords: Geometric mean; Contraharmonic mean; Inverse mean problem; Nonlinear matrix equation; Positive definite matrix

1. Introduction

In [2] Anderson and coauthors have introduced a natural matrix generalization of the contraharmonic mean of scalars $(a^2 + b^2)/(a + b)$ and studied related fixed point and least squares problems on positive semidefinite matrices. The contraharmonic mean $C(A, B)$ of positive definite matrices A and B is defined by

E-mail address: ylim@knu.ac.kr

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2005.06.013

$$C(A, B) = A + B - 2(A^{-1} + B^{-1})^{-1}.$$

Inverse mean problems involving the contraharmonic mean are considered and answered for the problem of contraharmonic and arithmetic means in [2] where the inverse mean problem of contraharmonic and geometric means is proposed even for the scalar version. The geometric mean $A\#B$ of positive definite matrices A and B is defined by $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and is well-known in theory of matrix inequalities (cf. [4–6]) and appears in semidefinite programming (scaling point, [20]) and in geometry (geodesic middle, [16,8,18]). Although the inverse mean problem occurs on positive semidefinite matrices A and B, we restrict our attention to the positive definite case. Similar questions on geometric, arithmetic and harmonic means are answered in [3].

The inverse mean problem of contraharmonic and geometric means is to find positive definite matrices X and Y for the system of nonlinear matrix equations:

$$\begin{cases} A = C(X, Y), \\ B = X\#Y, \end{cases} \tag{1.1}$$

where A and B are given positive definite matrices of same size. The main result of this paper is that the inverse mean problem (1.1) is equivalent to a system of the well-known nonlinear matrix equations (Theorem 2.3):

$$\begin{cases} X = A + 2BX^{-1}B, \\ Y = X - BY^{-1}B. \end{cases} \tag{1.2}$$

The matrix equations $X = Q + A^*X^{-1}A$ and $X = Q - A^*X^{-1}A$ with Q positive definite have been studied recently by several authors (see [1,9,10]). For the application areas where the equations arise, see the references given in [1,12].

In Section 3, we compute an explicit solution T of the first equation

$$X = A + 2BX^{-1}B$$

of (1.2) and then assert that the inverse mean problem is equivalent to the matrix equation

$$X = T - BX^{-1}B.$$

Solving directly the matrix equation $X = T - BX^{-1}B$ we conclude that the inverse mean problem is solvable if and only if $2B \leqslant T$, equivalently $B \leqslant A$ (Theorem 3.3). Extreme solutions of the inverse mean problem are also discussed.

2. The inverse mean problem

The following characterization and properties of the geometric mean are well-known. See [4,5,13,18,7] for multivariable geometric means.

Lemma 2.1 (The Riccati Lemma). *The geometric mean $A\#B$ of positive definite matrices A and B is the unique positive definite solution of the Riccati equation*

$$XA^{-1}X = B.$$