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Strengthened existence and uniqueness conditions for search directions in semidefinite programming

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Abstract

Primal–dual interior-point (p–d i-p) methods for Semidefinite Programming (SDP) are generally based on solving a system of matrix equations for a Newton type search direction for a symmetrization of the optimality conditions. These search directions followed the derivation of similar p–d i-p methods for linear programming (LP). Among these, a computationally interesting search direction is the AHO direction. However, in contrast to the LP case, existence and uniqueness of the AHO search direction is not guaranteed under the standard nondegeneracy assumptions. Two different sufficient conditions are known that guarantee the existence and uniqueness independent of the specific linear constraints. The first is given by Shida–Shindoh–Kojima and is based on the semidefiniteness of the symmetrization of the product SX at the current iterate. The second is a centrality condition given first by Monteiro–Zanjácomo and then improved by Monteiro–Todd.

In this paper, we revisit and strengthen both of the above mentioned sufficient conditions. We include characterizations for existence and uniqueness in the matrix equations arising

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from the linearization of the optimality conditions. As well, we present new results on the relationship between the Kronecker product and the *symmetric Kronecker product* that arise from these matrix equations. We conclude with a proof that the existence and uniqueness of the AHO direction is a generic property for every SDP problem and extend the results to the general Monteiro–Zhang family of search directions.

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1. Introduction

Semidefinite Programming (SDP) has generated tremendous interest during the last 15 years, e.g. [31], both for the many applications and the mathematical elegance. Many of the early interesting algorithms were primal–dual interior-point (p–d i-p) methods based on solving a system of matrix equations for Newton type search directions. The search directions, after a symmetrization, followed the derivation of similar p–d i-p methods for linear programming (LP), mainly based on the hope to extend the computational and/or theoretical properties of successful p–d i-p methods for LP. However, in contrast to the LP case, existence and uniqueness of some of these search directions were not guaranteed under the standard nondegeneracy assumptions: (i) *the linear transformation $\mathcal{A}$ from the linear constraints is onto* and (ii) *Slater's constraint qualification* (*strict feasibility with respect to the cone constraints*) *holds for both primal and dual programs*. (We assume that condition (i) holds throughout this paper.)

There are many search directions proposed for p–d i-p methods for semidefinite programming. For an account up to 1997, see [11,26]. Some of the proposals describe a set of search directions such as Kojima, Shindoh and Hara [12] (KSH family), Zhang [32], Monteiro and Zhang [19] (denoted MZ family), Tunçel [29]. Another general approach to search directions is to compute the Gauss–Newton direction for the overdetermined optimality conditions, see Kruk et al. [13].

Among the earliest proposals were the HrvwKshM direction [9,12,15], NT direction [21], and the AHO direction [1]. For each value of the central path parameter (or the barrier parameter), $\mu > 0$, these search directions, under standard nondegeneracy assumptions, are found by solving a particular (symmetrized) linear system of matrix equations. The HrvwKshM and NT directions have the disadvantage that the linear system becomes ill-conditioned as μ approaches zero (it is singular for $\mu = 0$); while the linear system for the AHO direction is nonsingular at $\mu = 0$. On the other hand, the HrvwKshM and NT directions are well-defined for every pair of primal–dual interior-points, though the AHO direction is not.