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Algebraic tools for the study of quaternionic behavioral systems

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Abstract

In this paper we study behavioral systems whose trajectories are given as solutions of quaternionic difference equations. As happens in the commutative case, it turns out that quaternionic polynomial matrices play an important role in this context. Therefore we pay special attention to such matrices and derive new results concerning their Smith form. Based on these results, we obtain a characterization of system theoretic properties such as controllability and stability of a quaternionic behavior.

Keywords: Linear systems; Quaternions; Behavioral approach; Smith form

1. Introduction

The behavioral approach to dynamical systems, introduced by Willems [15,16] in the eighties, considers as the main object of study in a system the set of all the trajectories which are compatible with its laws, known as the system behavior. Whereas the classical approaches start by dividing the trajectories into input, output and/or state space variables, according to some predefined mathematical model (for instance, the

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input-output or the state space model), the point of view of the behavioral approach is rather innovative. One looks at the set of trajectories without imposing any structure, i.e., without speaking, at an early stage, of inputs and outputs, of causes and effects. This point of view does not only unify the previous approaches, fitting them within an elegant theory, but it also permits to study a larger class of dynamical systems, including situations where it is not possible or desirable to make any distinction between input and output variables.

During the last two decades the importance of Clifford algebras, and in particular of the quaternion algebra, has been widely recognized. Actually, using this algebra, phenomena occurring in areas such as electromagnetism, quantum physics and robotics may be described by a more compact notation [5,7].

Systems with quaternionic signals were already investigated in the classic state space approach [4]. Here we aim at laying the foundations of the theory of quaternionic systems in the behavioral approach. Although every quaternionic system can be regarded as a complex or real system of higher dimension with special structure, keeping at the quaternionic level (i.e., viewing it as a system over $\mathbb{H}$) allows higher efficiency in computational terms. Since quaternionic polynomial matrices play an important role in this context, a considerable part of our work is devoted to the study of such matrices and in particular to their Smith form. The obtained results are relevant for the algebraic characterization of system theoretic properties.

The structure of the paper is as follows. In Section 2, after presenting the quaternionic skew-field $\mathbb{H}$ and quaternionic matrices, we refer to some examples that show the advantages of using quaternions in the description and solution of well-known physical problems. In Section 3 we introduce basic notions of quaternionic behavioral theory and show how to extend usual concepts of commutative linear algebra to the quaternionic algebra. Then, in Section 4, we characterize the Smith form of complex adjoint matrices and make its relation to the quaternionic Smith form explicit. Sections 5 and 6 are concerned with the characterization of controllability and stability of quaternionic behaviors.

2. Quaternions and applications

Let $\mathbb{R}$ denote the field of real numbers. The quaternion skew-field $\mathbb{H}$ is an associative but noncommutative algebra over $\mathbb{R}$ defined as the set

$$\mathbb{H} = \{a + bi + cj + dk \,:\, a, b, c, d \in \mathbb{R}\},$$

where i, j, k are called imaginary units and satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

This implies that

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$