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Representations of the Drazin inverse for a class of block matrices

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Abstract

In this paper we give a formula for the Drazin inverse of a block matrix $F = \begin{pmatrix} I & I \\ E & 0 \end{pmatrix}$, where E is square and I is the identity matrix. Further, we get representations of the Drazin inverse for matrices of the form $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where A is square, under some conditions expressed in terms of the individual blocks.

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1. Introduction

Our first purpose in this paper is to present a formula for the Drazin inverse of

$$F = \begin{pmatrix} I & I \\ E & 0 \end{pmatrix}, \quad E \text{ square and } I \text{ identity.} \tag{1.1}$$

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This research came up when we were interested in calculating the Drazin inverse of bordered matrices of the form $A = \begin{pmatrix} I & P^{\mathrm{t}} \\ Q & UV^{\mathrm{t}} \end{pmatrix}$, where U, V, P and Q are $n \times k$ matrices [4]. The matrix A can be written as $A = BC^{\mathrm{t}}$, where $B = \begin{pmatrix} 0 & I & I \\ U & Q & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & I & 0 \\ V & 0 & P \end{pmatrix}$. Hence, $A^{\mathrm{D}} = (BC^{\mathrm{t}})^{\mathrm{D}} = B((C^{\mathrm{t}}B)^{\mathrm{D}})^2C^{\mathrm{t}}$. Since

$$C^{\mathrm{t}}B = \begin{pmatrix} 0 & V^{\mathrm{t}} \\ I & 0 \\ 0 & P^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} 0 & I & I \\ U & Q & 0 \end{pmatrix} = \begin{pmatrix} V^{\mathrm{t}}U & V^{\mathrm{t}}Q & 0 \\ 0 & I & I \\ P^{\mathrm{t}}U & P^{\mathrm{t}}Q & 0 \end{pmatrix},$$

if $V^{\mathrm{t}}Q = 0$ or $P^{\mathrm{t}}U = 0$ we get a 2×2 block triangular matrix which second diagonal submatrix is $\begin{pmatrix} I & I \\ P^{\mathrm{t}}Q & 0 \end{pmatrix}$, which is of the kind (1.1). Following the representation of the Drazin inverse for block triangular matrices [10], we will get the Drazin inverse of $C^{\mathrm{t}}B$ in terms of the individual blocks. For that, we needed to find the Drazin inverse of matrices of the form of F. The Drazin inverse of a bordered matrix will be examined in a forthcoming specific work.

Our second purpose is to give a representation for the Drazin inverse of a block matrix of the form

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad A \text{ square and } 0 \text{ square null matrix,}$$

under some conditions expressed in terms of the individual blocks. Remark that B and C are rectangular matrices. Block matrices of this form arise in numerous applications, ranging from constrained optimization problems (KKT linear systems) to the solution of differential equations [2,6,9].

At the present time there is no known representation for the Drazin inverse of M with arbitrary blocks. This is an open research problem proposed by S.L. Campbell in [1], in the context of second order systems of differential equations.

Recently, in [13] and [12, Theorem 5.3.8], expressions for the Drazin inverse of 2×2 block matrix have been given involving the generalized Schur complement under some conditions.

Let us recall that the Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^{\mathrm{D}} \in \mathbb{C}^{n\times n}$ satisfying the relations

$$A^{\mathrm{D}}AA^{\mathrm{D}} = A^{\mathrm{D}}, \quad AA^{\mathrm{D}} = A^{\mathrm{D}}A, \quad A^{l+1}A^{\mathrm{D}} = A^l \quad \text{for all } l \geqslant r, \tag{1.2}$$

where r is the smallest nonnegative integer such that $\mathrm{rank}(A^r) = \mathrm{rank}(A^{r+1})$, i.e., $r = \mathrm{ind}(A)$, the index of A. The case when $\mathrm{ind}(A) = 1$, the Drazin inverse is called the group inverse of A and is denoted by $A^{\sharp}$.