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Linear maps transforming the higher numerical ranges

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Abstract

Let $k \in \{1, \ldots, n\}$. The k-numerical range of $A \in M_n$ is the set

$$W_k(A) = \{(\operatorname{tr} X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\},$$

and the k-numerical radius of A is the quantity

$$w_k(A) = \max\{|z| : z \in W_k(A)\}.$$

Suppose $k > 1$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n, k)\min\{k', n' - k'\}$. It is shown that there is a linear map $\phi : M_n \to M_{n'}$ satisfying $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in M_n$ if and only if $n'/n = k'/k$ or $n'/n = k'/(n - k)$ is a positive integer. Moreover, if such a linear map ϕ exists, then there are unitary matrix $U \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that ϕ has the form

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^{\mathrm{t}} \oplus \cdots \oplus A^{\mathrm{t}}}_{q}]U$$

or

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^{\mathrm{t}} \oplus \cdots \oplus \psi(A)^{\mathrm{t}}}_{q}]U,$$

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where $\psi : M_n \to M_n$ has the form $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$. Linear maps $\tilde{\phi} : M_n \to M_{n'}$ satisfying $w_{k'}(\tilde{\phi}(A)) = w_k(A)$ for all $A \in M_n$ are also studied. Furthermore, results are extended to triangular matrices.

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1. Introduction

There has been a great deal of interest in studying linear operator $\phi : \mathscr{M} \to \mathscr{M}$, where $\mathscr{M}$ is a matrix algebra or space, with a certain special property such as:

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathscr{M}$, where f is a given function on $\mathscr{M}$;
(b) $\phi(\mathscr{S}) \subseteq \mathscr{S}$ or $\phi(\mathscr{S}) = \mathscr{S}$ for a certain subset $\mathscr{S} \subseteq \mathscr{M}$;
(c) $\phi(A) \sim \phi(B)$ in $\mathscr{M}$ whenever $A \sim B$ in $\mathscr{M}$ for a certain relation $\sim$ on $\mathscr{M}$.

Very often, ϕ has nice forms such as

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^{\mathrm{t}}N$$

for some suitable $M, N \in \mathscr{M}$. One may see [19] for a survey on the subject. Recently, there has been research on more general problems concerning linear transformations ϕ: $\mathscr{M} \to \mathscr{M}'$ with some special properties such as

(a) $f'(\phi(A)) = f(A)$ for all $A \in \mathscr{M}$, where f and f' are appropriate functions on $\mathscr{M}$ and $\mathscr{M}'$;
(b) $\phi(\mathscr{S}) \subseteq \mathscr{S}'$ or $\phi(\mathscr{S}) = \mathscr{S}'$ for certain subsets $\mathscr{S} \subseteq \mathscr{M}$ and $\mathscr{S}' \subseteq \mathscr{M}'$;
(c) $\phi(A) \sim' \phi(B)$ in $\mathscr{M}'$ whenever $A \sim B$ in $\mathscr{M}$ for certain relations $\sim$ on $\mathscr{M}$ and $\sim'$ on $\mathscr{M}'$.

Such problems are more challenging and their study often lead to the discovery of unexpected results and hidden structures of the matrix algebras $\mathscr{M}$ and $\mathscr{M}'$; see [6, 10]. In this paper, we consider these types of problems. We solve a specific problem and develop some proof techniques that may be useful for future study in this area.

Let us first introduce some notations and definitions. Denote by M_n the algebra of $n \times n$ complex matrices. For $1 \leqslant k \leqslant n$, define (see Halmos [11]) the k-numerical range of $A \in M_n$ as

$$W_k(A) = \{(\operatorname{tr} X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\}.$$

Since $W_n(A) = \{\operatorname{tr} A/n\}$, we always assume that $k < n$ to avoid trivial consideration. When $k = 1$, we have the classical numerical range $W_1(A)$, which is useful