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The spectral characterization of generalized projections ☆

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Abstract

In this note, the spectral characterization of generalized projections are established.

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Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the set of all linear bounded operators on $\mathscr{H}$. In this note, a further properties of generalized projections are studied.

Definition 1. $A \in \mathscr{B}(H)$ is said to be a generalized projection if $A^2 = A^*$, where T^* is the adjoint of T.

The notation of generalized projections on a finite dimensional Hilbert space introduced by Groß and Trenkler [4]. In this note, the concept of generalized projections is extended on $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is not necessarily finite dimensional. Groß and Trenkler in [4] have asserted that a square complex matrix K is a generalized

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projection if and only if (1) $K = K^4$, (2) K is normal ($KK^* = K^*K$) and (3) K is a partial isometry. Baksalary and Liu in [2] have verified that one of the conditions (2) and (3) can be deleted. In [1], the linear combinations of generalized projections have been established. In this note, the spectral characterizations of generalized projections are obtained by using spectral theory of operators (see [3,5]). The spectral characterization will lead us to understand of the geometry structure of generalized projections.

In this note, we will prove the following theorem.

Theorem 2. *Let $A \in \mathscr{B}(\mathscr{H})$. Then A is a generalized projection if and only if A is a normal operator and $\sigma(A) \subseteq \{0, 1, e^{\pm i\frac{2}{3}\pi}\}$. In this case, A has the following spectral representation*

$$A = 0E(0) \oplus E(1) \oplus e^{i\frac{2}{3}\pi} E(e^{i\frac{2}{3}\pi}) \oplus e^{-i\frac{2}{3}\pi} E(e^{-i\frac{2}{3}\pi}), \tag{1}$$

where $E(\alpha)$ denotes the spectral projection associated with a spectral point $\alpha \in \sigma(A)$ and $E(\alpha) = 0$ if $\alpha \notin \sigma(A)$.

To complete the proof, we first recall some notations. For an operator T, the range, the null space and the spectrum of T are denoted by $\mathscr{R}(T)$, $\mathscr{N}(T)$ and $\sigma(T)$, respectively. An operator $N \in \mathscr{B}(\mathscr{H})$ is said to be normal if $N^*N = NN^*$. If N is a normal operator, then there exists a unique spectral measure E on the Borrel subsets of $\sigma(N)$ such that N has the following spectral representation (see [3])

$$N = \int_{\sigma(N)} \lambda \, dE_\lambda.$$

An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be idempotent if $A^2 = A$. An operator $P \in \mathscr{B}(\mathscr{H})$ is said to be an orthogonal projection if $P^2 = P = P^*$.

Proof of Theorem 2. If $A^2 = A^*$, then

$$AA^* = A^3 = A^2A = A^*A,$$

this shows that A is normal.

Let $A = \int_{\sigma(A)} \lambda \, dE_\lambda$ be the spectral representation of A, then $A^* = \int_{\sigma(A)} \overline{\lambda} \, dE_\lambda$. By the assumption of that A is a generalized projection, $A^2 = A^*$, we have

$$A^2 - A^* = \int_{\sigma(A)} (\lambda^2 - \overline{\lambda}) \, dE_\lambda = 0.$$

Hence, $\lambda^2 = \overline{\lambda}$ for all $\lambda \in \sigma(A)$. If $\lambda \in \sigma(A)$ and $\lambda \neq 0$, and denote $\lambda = re^{i\theta}$ with $-\pi < \theta \leqslant \pi$, then $r^2e^{2i\theta} = re^{-i\theta}$ and $r \neq 0$, so

$$re^{i\theta} = e^{-2i\theta}.$$