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Eigenstructure of order-one-quasiseparable matrices. Three-term and two-term recurrence relations

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Abstract

This paper presents explicit formulas and algorithms to compute the eigenvalues and eigenvectors of *order-one-quasiseparable matrices*. Various recursive relations for characteristic polynomials of their principal submatrices are derived. The cost of evaluating the characteristic polynomial of an $N \times N$ matrix and its derivative is only $O(N)$. This leads immediately to several versions of a fast *quasiseparable Newton iteration* algorithm. In the Hermitian case we extend the Sturm property to the characteristic polynomials of order-one-quasiseparable matrices which yields to several versions of a fast *quasiseparable bisection algorithm*.

Conditions guaranteeing that an eigenvalue of a order-one-quasiseparable matrix is simple are obtained, and an explicit formula for the corresponding eigenvector is derived. The method is further extended to the case when these conditions are not fulfilled. Several particular examples with tridiagonal, (almost) unitary Hessenberg, and Toeplitz matrices are considered.

The algorithms are based on new three-term and two-term recurrence relations for the characteristic polynomials of principal submatrices of *order-one-quasiseparable matrices* R.

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It turns out that the latter *new class of polynomials* generalizes and includes two classical families: (i) polynomials orthogonal on the real line (that play a crucial role in a number of classical algorithms in numerical linear algebra), and (ii) the Szegö polynomials (that play a significant role in signal processing). Moreover, new formulas can be seen as generalizations of the classical three-term recurrence relations for the real orthogonal polynomials and of the two-term recurrence relations for the Szegö polynomials.

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1. Introduction

1.1. Quasiseparable and diagonal-plus-semiseparable matrices

Following [8] we refer to R as an *order*-$(r_{\mathrm{L}}, r_{\mathrm{U}})$-*quasiseparable* matrix if

$$r_{\mathrm{L}} = \max \operatorname{rank} R_{21}, \quad r_{\mathrm{U}} = \max \operatorname{rank} R_{12}, \tag{1.1}$$

where the maximum is taken over all *symmetric* partitions of the form $R = \left[\begin{array}{c|c} * & R_{12} \\ \hline R_{21} & * \end{array}\right]$. In case $r_{\mathrm{U}} = r_{\mathrm{L}} = r$ one refers to R as an order-r-quasiseparable matrix. Quasiseparable matrices generalize *diagonal-plus-semiseparable matrices* [17], i.e., those of the form

$$R = \operatorname{diag}(d) + \operatorname{tril}(R_{\mathrm{L}}) + \operatorname{triu}(R_{\mathrm{U}}), \quad \text{where } \operatorname{rank} R_{\mathrm{L}} = r_{\mathrm{L}}, \ \operatorname{rank} R_{\mathrm{U}} = r_{\mathrm{U}} \tag{1.2}$$

with some vector d, and with some matrices R_{L}, R_{U}. Here $\operatorname{tril}(R)$ and $\operatorname{triu}(R)$ are the standard MATLAB notations standing for the strictly lower and strictly upper triangular parts of R, resp. Clearly, diagonal-plus-semiseparable matrices are quasiseparable, but not vice versa, e.g., the tridiagonal matrix in (1.5) is not diagonal-plus-semiseparable. Quasiseparable matrices are also known under different names such as matrices with a low Hankel rank [6], low mosaic rank matrices or weakly semiseparable matrices [22]. In the context of some applications, e.g., to integral equations, electromagnetics, boundary value problems, the less general diagonal-plus-semiseparable matrices occur more often. Moreover, the orders higher than $(1, 1)$ have attracted more attention perhaps since they exhibit somewhat more computational challenges.

The next section suggests that even in the simplest case of the order $(1, 1)$ the more general quasiseparable matrices are by no means less interesting. Moreover,