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A Perron Theorem for positive componentwise bilinear maps

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Abstract

It is proved that a Perron type theorem holds for positive maps with bilinear components whose defining matrices satisfy a maximality assumption with respect to certain entry ratios. The result is applied to a life history model which includes sexual reproduction.

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1. Introduction

The common proof of the Perron–Frobenius Theorem, which gives insights into the behavior of the successive iterates of any positive square matrix as a function on the positive real cone of corresponding dimension, is an algebraic one. However, a large part of this theorem can be readily proved analytically using the theorem of Birkhoff, [1] and [2–pp. 383–385], which states that such a matrix induces a contraction mapping on the projective quotient of the cone with respect to the Hilbert

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projective metric. With coauthors, Kohlberg [3,4–and unpublished work with Pratt] has further investigated this metric. In addition, Kohlberg [5] and other investigators, e.g., [6], have established that some functions on the positive cone which share certain properties with positive matrices, including nonnegativity and homogeneity of degree one, are contraction mappings in the same sense and, therefore, have associated Perron theorems. Here, we study positive mappings defined on the product of two positive cones with bilinear vector component functions, and their compositions with some homogeneous maps on each of the two vector coordinates separately. As we shall see, such mappings can be useful in the description of life histories of some simple populations which reproduce sexually.

2. The basic bilinear model

The motivation for this paper is a reproductive model. We consider a two sex population with m (pheno)types of females and n types of males, and assume type inheritance as follows. For $i = 1, \dots, m$ and $j = 1, \dots, n$, let for $k = 1, \dots, m$, $\sigma_{kij} \geqslant 0$ be the proportion of female offspring from a mating between a type i female and a type j male which are of type k and, for $k = 1, \dots, n$, $\tau_{kij} \geqslant 0$ be the proportion of male offspring from the same mating which are of type k. Phenotypic inheritance of this sort might occur for traits determined by multiple genes or might be simply all that is observable if the genotypes which determine the phenotypes are unknown. Clearly, for any (i, j), $\sum_{k=1}^{m} \sigma_{kij} = 1 = \sum_{k=1}^{n} \tau_{kij}$. For $k = 1, \dots, m$, let $\mathbf{S}_k$ be the $m \times n$ matrix whose (i, j)th term is σ_{kij} and for $k = 1, \dots, n$, let $\mathbf{T}_k$ be the $m \times n$ matrix whose (i, j)th term is τ_{kij}. Then $\sum_{k=1}^{m} \mathbf{S}_k = \sum_{k=1}^{n} \mathbf{T}_k$ is the constant $m \times n$ matrix with all entries 1.

For any positive integer m, let $\mathbf{R}_+^m = \{(x_1, \dots, x_m) \in \mathbf{R}^m - \{\mathbf{0}\} \mid x_i \geqslant 0$ for $i = 1, \dots, m\}$, the nonnegative cone in $\mathbf{R}^m$. Let $|\cdot|$ denote the ℓ_1 norm on $\mathbf{R}^m$, i.e., $|\mathbf{x}| = |x_1| + \cdots + |x_m|$, and observe that for $\mathbf{x} \in \mathbf{R}_+^m$, $|\mathbf{x}| = x_1 + \cdots + x_m$. Let $\mathbf{H}_+^{m-1} = \{\mathbf{x} \in \mathbf{R}_+^m \mid |\mathbf{x}| = 1\}$, the space of nonnegative stochastic vectors in $\mathbf{R}^m$, a compact subset of an $m - 1$ dimensional hyperplane. Initially, we shall track the phenotypic proportions, rather than the absolute numbers, of the female and male populations. Then, female vectors lie in $\mathbf{H}_+^{m-1}$ and male vectors in $\mathbf{H}_+^{n-1}$, so the space of population vectors is $\mathscr{H}_+ = \mathbf{H}_+^{m-1} \times \mathbf{H}_+^{n-1}$. Mating is assumed to be random and the offspring become the next generation. Therefore, if $\mathbf{x}(t)$ represents the female vector of the tth generation and $\mathbf{y}(t)$ the male vector, the transformation on $\mathscr{H}_+$ which carries one generation to the next uses the $\mathbf{S}_k$ and $\mathbf{T}_k$ as bilinear forms, namely

$$x_k(t+1) = \mathbf{x}(t)^{\mathrm{T}} \mathbf{S}_k \mathbf{y}(t), \quad k = 1, \dots, m, \tag{1a}$$

$$y_k(t+1) = \mathbf{x}(t)^{\mathrm{T}} \mathbf{T}_k \mathbf{y}(t), \quad k = 1, \dots, n. \tag{1b}$$

The summation conditions on the $\mathbf{S}_k$'s and $\mathbf{T}_k$'s imply that $(\mathbf{x}(t+1), \mathbf{y}(t+1)) \in \mathscr{H}_+$, as we wish, so our model is that of a discrete dynamical system on $\mathscr{H}_+$. Let