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Schur complements and state space realizations[☆]

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Abstract

Motivated by state space realizations of transfer functions from system theory, a number of operations on Schur complements are introduced and studied. These operations are equivalence, extension, multiplication, inversion, and factorization. Together they form an algebraic framework which is of independent interest, and also useful in solving problems in analysis.

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0. Introduction

Throughout this paper S will be a (2×2) operator (matrix) of the type

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \dot{+} Y \to \widehat{X} \dot{+} \widehat{Y}, \tag{1}$$

where X, Y, $\widehat{X}$ and $\widehat{Y}$ are complex Banach spaces. Suppose $A : X \to \widehat{X}$ is invertible. Then, by (block) Gauss elimination,

$$S = \begin{bmatrix} I_{\widehat{X}} & 0 \\ CA^{-1} & I_{\widehat{Y}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_X & A^{-1}B \\ 0 & I_Y \end{bmatrix}, \tag{2}$$

where the first factor and the last factor in the right hand side are both invertible. The second term in the diagonal of the factor in the middle of (2),

$$W_1(S) = W_1\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = D - CA^{-1}B : Y \to \widehat{Y}$$

is called the *first Schur complement in* S. Other names that are in vogue are *Schur complement of* A *in* S and *Schur complement of* S *relative to* A; also instead of $W_1(S)$, one finds the notation S/A (cf., [20, Chapter 1]).

Analogously, whenever $D : Y \to \widehat{Y}$ is invertible,

$$S = \begin{bmatrix} I_{\widehat{X}} & BD^{-1} \\ 0 & I_{\widehat{Y}} \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_X & 0 \\ D^{-1}C & I_Y \end{bmatrix}, \tag{3}$$

and the operator

$$W_2(S) = W_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = A - BD^{-1}C : X \to \widehat{X}$$

is said to be the *second Schur complement in* S. Clearly, in this situation,

$$W_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = W_1\left(\begin{bmatrix} D & C \\ B & A \end{bmatrix}\right), \tag{4}$$

and hence every result for first Schur complements has a counterpart for second Schur complements and vice versa.

Schur complements arise naturally in mathematical system theory. Indeed, when in (1) the spaces X and $\widehat{X}$ coincide and the operator A is replaced with $A - \lambda$ (shorthand for $A - \lambda I_X$), one has

$$W_1\left(\begin{bmatrix} A - \lambda & B \\ C & D \end{bmatrix}\right) = D + C(\lambda - A)^{-1}B. \tag{5}$$

The right hand side of this expression is a state space realization of the transfer function of the linear time invariant system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ y(t) = Cx(t) + Du(t), & t \geqslant 0, \\ x(0) = 0. \end{cases} \tag{6}$$