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On similarity of matrices over commutative rings

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Abstract

The paper studies the problem on matrix similarity over a commutative rings. The conditions are determined, under which the matrix is similar to a companion or diagonal matrices.

Keywords: Matrices; Eigenvalue; Eigenvector; Companion matrix

Let R be a commutative ring with the unit element $e \neq 0$ and $U(R)$ the set of divisors of unit element e. Further, let $R_{n,m}$ and $R_{n,m}[x]$ be the sets of $n \times m$ matrices over R and the polynomial ring $R[x]$, respectively. Denote by I and O, respectively, the identity and zero matrices of order n. Let $b(x) = x^k - b_1x^{k-1} - b_2x^{k-2} - \cdots - b_{k-1}x - b_k$ be a monic polynomial of degree k in $R[x]$. The $k \times k$ matrix

$$F_b = \begin{Vmatrix} 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \dots & \dots & 0 & e \\ b_k & b_{k-1} & \dots & b_2 & b_1 \end{Vmatrix}$$

is called the companion matrix of $b(x)$.

Recall two matrices A and B in $R_{n,n}$ are said to be similar, if $A = TBT^{-1}$ for some matrix $T \in GL(n, R)$. Let $A \in R_{n,n}$ be a matrix with a characteristic polynomial

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$$\det(Ix - A) = a(x) = x^n - a_1x^{n-1} - a_2x^{n-2} - \cdots - a_{n-1}x - a_n.$$

The aim of the present paper is to find the conditions of similarity of matrices A and F_a. In case, if $\det(Ix - A) = \prod_{k=1}^{n}(x - \alpha_k)$, we establish conditions for existence of matrix $T \in GL(n, R)$ such that $TAT^{-1} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R_{n,n}$—a diagonal matrix. For the case, where R is a field or a ring, these problems were studied in [1–7].

Proposition 1. *Let the characteristic polynomial of a matrix $A \in R_{n,n}$ admit the representation*

$$\det(Ix - A) = d_1(x)d_2(x)\ldots d_m(x),$$

where $d_i(x) = x^{k_i} - d_{i,1}x^{k_i-1} - \cdots - d_{i,k_i-1}x - d_{i,k_i}$ – are monic polynomials over R, $\deg d_i(x) = k_i$, $1 \leqslant k_i \leqslant n$, $i = 1, 2, \ldots, m$; $k_1 + k_2 + \cdots + k_m = n$. If the matrix A is similar to a block diagonal matrix $D_A = \operatorname{diag}(F_{d_1}, F_{d_2}, \ldots, F_{d_m})$, then for every matrix $T \in GL(n, R)$ such that $TAT^{-1} = D_A$, there exist rows $t_1, t_2, \ldots, t_m \in R_{1,n}$ such that

$$T = \left\| \begin{matrix} T_1 \\ T_2 \\ \vdots \\ T_m \end{matrix} \right\|, \quad \text{where } T_i = \left\| \begin{matrix} t_i \\ t_i A \\ \vdots \\ t_i A^{k_i-2} \\ t_i A^{k_i-1} \end{matrix} \right\|, \ i = 1, 2, \ldots, m.$$

Proof. Let $T \in GL(n, R)$ be such matrix that

$$TAT^{-1} = D_A = \operatorname{diag}(F_{d_1}, F_{d_2}, \ldots, F_{d_m}). \tag{1}$$

Note that T can be rewritten as

$$T = \left\| \begin{matrix} T_1 \\ \vdots \\ T_i \\ \vdots \\ T_m \end{matrix} \right\|, \quad \text{where } T_i = \left\| \begin{matrix} \bar{t}_{i,1} \\ \bar{t}_{i,2} \\ \vdots \\ \bar{t}_{i,k_i-1} \\ \bar{t}_{i,k_i} \end{matrix} \right\| \in R_{k_i,n}; \ \bar{t}_{i,j} \in R_{1,n};$$

$j = 1, 2, \ldots, k_i$; $i = 1, 2, \ldots, m$. From equality (1) we find that $T_i A = \left\| O_1 \quad F_{d_i} \quad O_2 \right\| T$, where O_1 and O_2 are zero $(k_i \times (k_1 + \cdots + k_{i-1}))$ and $(k_i \times (k_{i+1} + \cdots + k_m))$ matrices respectively. From the last equality we conclude that

$$\begin{cases} \bar{t}_{i,1}A = \bar{t}_{i,2}, \\ \bar{t}_{i,2}A = \bar{t}_{i,1}A^2 = \bar{t}_{i,3}, \\ \cdots \quad \cdots \quad \cdots \\ \bar{t}_{i,k_i-2}A = \bar{t}_{i,1}A^{k_i-2} = \bar{t}_{i,k_i-1}, \\ \bar{t}_{i,k_i-1}A = \bar{t}_{i,1}A^{k_i-1} = \bar{t}_{i,k_i}. \end{cases}$$