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Perron–Frobenius theory of seminorms: a topological approach ☆

Assaf Goldberger [a], Michael Neumann [b,*]

[a] School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel
[b] Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

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Abstract

For nonnegative matrices A, the well known Perron–Frobenius theory studies the spectral radius $\rho(A)$. Rump has offered a way to generalize the theory to arbitrary complex matrices. He replaced the usual eigenvalue problem with the equation $|Ax| = \lambda|x|$ and he replaced $\rho(A)$ by the *signed spectral radius*, which is the maximum λ that admits a nontrivial solution to that equation. We generalize this notion by replacing the linear transformation A by a map $f : \mathbb{C}^n \to \mathbb{R}$ whose coordinates are seminorms, and we use the same definition of Rump for the signed spectral radius. Many of the features of the Perron–Frobenius theory remain true in this setting. At the center of our discussion there is an *alternative theorem* relating the inequalities $f(x) \geqslant \lambda|x|$ and $f(x) < \lambda|x|$, which follows from topological principals. This enables us to free the theory from matrix theoretic considerations and discuss it in the generality of seminorms. Some consequences for P-matrices and D-stable matrices are discussed.

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* Corresponding author.
E-mail address: neumann@math.uconn.edu (M. Neumann).

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1. Introduction

Let A be nonnegative matrix. In the Perron–Frobenius theory one is interested in the dominant eigenvalue $\rho = \rho(A)$ of A. Some well known properties of ρ are: (i) ρ is a nonnegative number which corresponds to a nonnegative eigenvector. (ii) ρ is the largest number λ, such that there exists a nonzero vector v with $|Av| \geqslant \lambda|v|$. Here and throughout, absolute values and inequalities of vectors are taken componentwise. (iii) If B is a principal submatrix of A, denoted by $B \preceq A$, then $\rho(B) \leqslant \rho(A)$. (iv) If $\rho(A)$ is strictly bigger than $\rho(B)$ for all $B \preceq A$, then ρ corresponds to a strictly positive eigenvector of A and it is a simple eigenvalue. (v) $\rho = 0$ if and only if A is permutationally similar to an upper triangular nilpotent matrix. (vi) There are min–max formulas that characterize ρ via the Collatz–Wielandt ratios.

Suppose now that A is a general square matrix over $F = \mathbb{R}$ or $\mathbb{C}$. Rump [4,5,6] has defined the *signed spectral radius* of A as

$$\rho^F(A) := \max_S \rho_0^F(SA), \tag{1.1}$$

where S runs through the set of all the signature matrices over F and where $\rho_0^F(A)$ is the maximal eigenvalue over F. In the case where $F = \mathbb{R}$ and where there is no real eigenvalue we set $\rho_0^F(A) = 0$. Rump shows that $\rho^F(A)$ as defined above satisfies similar properties to (i)–(vi). There are a few differences that result from the fact that A is no longer nonnegative. The main difference is that now there is no generic orthant where the vector that corresponds to the maximal eigenvector occurs. In this setting, it turns out that properly (iv) is replaced by the following property: If $\rho^F(A)$ is strictly bigger than $\rho^F(B)$ for all $B \preceq A$, then the eigenvalue $\rho = \rho^F(A)$ of *some* SA belongs to a vector whose coordinates are nonzero, and that it is the only eigenvalue that corresponds to a vector in the same orthant. The min–max formulas also differ slightly from those mentioned in (vi), since one has to maximize over all the orthants. An interesting point is that when A is a real matrix, $\rho^{\mathbb{R}}(A) \leqslant \rho^{\mathbb{C}}(A)$ and the quantities need not be equal. Thus one has two distinct spectral radii that behave like the Perron root. When $A \geqslant 0$, then $\rho^{\mathbb{R}}(A) = \rho^{\mathbb{C}}(A) = \rho(A)$ and the theory developed by Rump reduces the the usual Perron–Frobenius theory.

Initially our work was motivated by a conjecture of Rump, which claims that for square real matrices A with 1's along the main shift-cycle, there exists a vector $x \neq 0$ such that $|Ax| \geqslant \frac{1}{2}|x|$. (Originally we looked at another version of this conjecture that was proven to be false.) The main question that concerned us was how to prove the existence of such a vector x. Our thought was that the existence of x should follow from topological principles, in the same way that Brouwer's fixed-point theorem is an outcome of topological principles. This turned out to be true and we were able to prove the following theorem, which we shall call the *Special Alternative Theorem.*

Theorem 1.1. *Let $A \in M_n(F)$ be any matrix and let $\lambda \geqslant 0$. Then precisely one of the following two conditions must hold*: