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Tridiagonal pairs and the Askey–Wilson relations

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Abstract

The notion of a tridiagonal pair was introduced by Ito, Tanabe and Terwilliger. Let V denote a nonzero finite dimensional vector space over a field $\mathscr{F}$. A tridiagonal pair on V is a pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy some conditions. Assume (A, A^*) is a tridiagonal pair on V. Recently Terwilliger and Vidunas showed that if A is multiplicity-free on V, then (A, A^*) satisfy the following "Askey–Wilson relation" for some scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$.

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I,$$
$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I.$$

In the paper, we show that, if a tridiagonal pair (A, A^*) satisfy the Askey–Wilson relations, then the eigenspaces of A and the eigenspaces of A^* have one common dimension, and moreover if $\mathscr{F}$ is algebraically closed then that common dimension is 1.

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1. Introduction

The notion of a tridiagonal pair was introduced by Ito, Tanabe and Terwilliger, that is defined as follows.

Definition 1.1 [1]. Let V denote a nonzero finite dimensional vector space over a field $\mathscr{F}$. By a *tridiagonal pair* on V, we mean a pair (A, A^*), where $A : V \longrightarrow V$ and $A^* : V \longrightarrow V$ are linear transformations that satisfy the following conditions.

(i) A and A^* are both diagonalizable on V.
(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

Remark 1.2. With reference to Definition 1.1, it is known that $d = \delta$ and $\dim V_i = \dim V_i^*$ $(0 \leqslant i \leqslant d)$ [1–Lemma 4.5 and Corollary 5.7].

Throughout this paper, let (A, A^*) denote a tridiagonal pair on a vector space V over a field $\mathscr{F}$.

The notion of a Leonard pair was introduced by Terwilliger [3]. A Leonard pair (A, A^*) coincides with a tridiagonal pair such that each eigenspace of A is one dimensional. Recently Terwilliger and Vidunas [4–Theorem 1.5] showed that every Leonard pair (A, A^*) satisfies the following relations for some scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$.

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I, \tag{1}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I. \tag{2}$$

These relations are the *Askey–Wilson relations*, which was introduced by Zhedanov [5].

In the present paper, we prove the following results.

Theorem 1.3. *Suppose*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = p(A) \tag{3}$$