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Siegel transformations for even characteristic

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Abstract

Let V be a vector space over a field K of even characteristic and $|K| > 3$. Suppose K is perfect and π is an element in the special orthogonal group $SO(V)$ with $\dim B(\pi) = 2d$. Then $\pi = \rho_1 \cdots \rho_{d-1}\kappa$, where ρ_j, $j = 1, \dots, d-1$, are Siegel transformations and $\kappa \in SO(V)$ with $\dim B(\kappa) = 2$. The length of π with respect to the Siegel transformations is d if π is unipotent or if $\dim B(\pi)/\mathrm{rad}\, B(\pi) \geqslant 4$; otherwise it is $d + 1$.

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1. Introduction

Let G be a group and let S be a set of generators for G. Let π be an element in G, then $\pi = s_1 \cdots s_k$, where all s_j, $j = 1, \dots, k$, are elements in S. The *length* $\ell(\pi)$ of π with respect to S is the minimal k for which such a factorization exists. For certain groups G and certain generating systems S it is possible to determine $\ell(\pi)$ for each π in G.

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Bachmann [1] coined the phrase *length problem* for the program described above. The length problem for the orthogonal groups over fields of characteristic not 2 was solved by Scherk [12]. Further, Dieudonné [3] solved the length problem for several other classical groups. For more references see e.g. Ellers [4,5].

If Q is a nondegenerate singular quadratic form on a vector space V over a field K, then the commutator subgroup $G = \Omega(V)$ of $O(V)$ is generated by the set of Siegel transformations. Assuming that the field K of coefficients has characteristic not 2, Knüppel solved the length problem for $G = \Omega(V)$, where the generating set S is the set of Siegel transformations [10]. In the present paper, we assume that the characteristic of K is even, $|K| > 3$, and K is perfect. Under these conditions we solve the length problem for $\Omega(V)$ with respect to Siegel transformations.

In Section 3, we are laying the groundwork. Here we assume that V is nonsingular and that V contains singular vectors distinct from zero. We see that some of the properties established in [10] for orthogonal groups over fields K of characteristic not 2 are also valid when the characteristic of K is even.

In Section 4, we establish a lower bound for the Siegel length of an isometry. In Section 5, we assume that K is perfect, and we determine the Siegel length of an isometry π, Theorem 5.5. Here our approach differs entirely from that in [10]. Our tools include the factorization of an orthogonal transformation π into a product of two involutions [6,7] and also the factorization of π into a product $\pi = \mu \cdot \nu$, where μ is unipotent and the path of ν is nonsingular [8].

2. Notation

Let V be a vector space of dimension n over a field K where $|K| > 2$, equipped with a *quadratic form* Q (see [2–§16]), defined by $Q(\alpha v) = \alpha^2 Q(v)$ and $Q(v + w) = Q(v) + Q(w) + f(v, w)$ for some bilinear form f, where $\alpha \in K$ and $v, w \in V$. Two vectors $v, w \in V$ are called *perpendicular*, $v \perp w$, if $f(v, w) = 0$. A vector $v \in V$ is called *isotropic* if $f(v, v) = 0$ and *singular* if $Q(v) = 0$. Let W be a subspace of V. Then W is called *totally isotropic* if $f(u, w) = 0$ for all $u, w \in W$ and *totally singular* if $Q(w) = 0$ for all $w \in W$. A totally singular subspace is also totally isotropic, but the converse is not necessarily true. The subspaces $\operatorname{rad} W = W \cap W^{\perp}$ and $SW = \{x \in \operatorname{rad} W \mid Q(x) = 0\}$ are called the *radical* of W and the *singular* of W, respectively. The space W is said to be *nonsingular* if $\operatorname{rad} W = 0$.

The orthogonal group on V, denoted $O(V)$, is the set of isometries, i.e. of all transformations that preserve the value of Q. For $\pi \in O(V)$ we define $B(\pi) := V(\pi - 1)$ and $F(\pi) := \ker(\pi - 1)$. The subspaces $B(\pi)$ and $F(\pi)$ of V are called *path* and *fixed space* of π, respectively.

We shall always assume that V is nonsingular and that there is at least one $v \in V \setminus \{0\}$ such that $Q(v) = 0$.

We shall state a number of facts (see e.g. [13]).