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A note on eigenvalues of perturbed Hermitian matrices

Chi-Kwong Li [a,1], Ren-Cang Li [b,*,2]

[a] *Department of Mathematics, The College of William and Mary, Williamsburg, VA 23185, USA*
[b] *Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506, USA*

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Abstract

Let

$$A = \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}$$

be Hermitian matrices with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_k$ and $\widetilde{\lambda}_1 \geqslant \cdots \geqslant \widetilde{\lambda}_k$, respectively. Denote by $\|E\|$ the spectral norm of the matrix E, and η the spectral gap between the spectra of H_1 and H_2. It is shown that

$$|\lambda_i - \widetilde{\lambda}_i| \leqslant \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}},$$

which improves all the existing results. Similar bounds are obtained for singular values of matrices under block perturbations.

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* Corresponding author.
E-mail addresses: ckli@math.wm.edu (C.-K. Li), rcli@ms.uky.edu (R.-C. Li).
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1. Introduction

Consider a partitioned Hermitian matrix

$$A = \begin{array}{c} \\ m \\ n \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} H_1 & E^* \\ E & H_2 \end{pmatrix} \end{array}, \tag{1.1}$$

where E^* is E's complex conjugate transpose. At various situations (typically when E is *small*), one is interested in knowing the impact of removing E and E^* on the eigenvalues of A. More specifically, one would like to obtain bounds for the differences between that eigenvalues of A and those of its perturbed matrix

$$\widetilde{A} = \begin{array}{c} \\ m \\ n \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix} \end{array}. \tag{1.2}$$

Let $\lambda(X)$ be the spectrum of the square matrix X, and let $\|Y\|$ be the spectral norm of a matrix Y, i.e., the largest singular value of Y. There are two kinds of bounds for the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{m+n}$ and $\widetilde{\lambda}_1 \geqslant \cdots \geqslant \widetilde{\lambda}_{m+n}$ of A and $\widetilde{A}$, respectively:

(1) [1,7,8]

$$|\lambda_i - \widetilde{\lambda}_i| \leqslant \|E\|. \tag{1.3}$$

(2) [1–4,7,8] If the spectra of H_1 and H_2 are disjoint, then

$$|\lambda_i - \widetilde{\lambda}_i| \leqslant \|E\|^2/\eta, \tag{1.4}$$

where

$$\eta \stackrel{\text{def}}{=} \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} |\mu_1 - \mu_2|,$$

and $\lambda(H_i)$ is the spectrum of H_i.

The bounds of the first kind do not use information of the spectral distribution of the H_1 and H_2, which will give (much) weaker bounds when η is not so small; while the bounds of the second kind may blow up whenever H_1 and H_2 have a common eigenvalue. Thus both kinds have their own drawbacks, and it would be advantageous to have bounds that are always no bigger than $\|E\|$, of $\mathcal{O}(\|E\|)$ as $\eta \to 0$, and at the same time behave like $\mathcal{O}(\|E\|^2/\eta)$ for not so small η. To further motivate our study, let us look at the following 2×2 example.

Example 1. Consider the 2×2 Hermitian matrix

$$A = \begin{pmatrix} \alpha & \epsilon \\ \epsilon & \beta \end{pmatrix}. \tag{1.5}$$