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Spectral theory of copositive matrices

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Abstract

Let $A \in \mathbf{R}^{n\times n}$. We provide a block characterization of copositive matrices, with the assumption that one of the principal blocks is positive definite. Haynsworth and Hoffman showed that if r is the largest eigenvalue of a copositive matrix then $r \geqslant |\lambda|$, for all other eigenvalues λ of A. We continue their study of the spectral theory of copositive matrices and show that a copositive matrix must have a positive vector in the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues. Moreover, if a symmetric matrix has a positive vector in the subspace spanned by the eigenvectors corresponding to its nonnegative eigenvalues, then it is possible to increase the nonnegative eigenvalues to form a copositive matrix A', without changing the eigenvectors. We also show that if a copositive matrix has just one positive eigenvalue, and $n-1$ nonpositive eigenvalues then A has a nonnegative eigenvector corresponding to a nonnegative eigenvalue.

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1. Introduction

Let $e_i \in \mathbf{R}^n$ denote the vector with a 1 in the ith position and all 0's elsewhere. For $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$ we will use the notation that $x \geqslant 0$ when $x_i \geqslant 0$ for

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all i, $1 \leqslant i \leqslant n$, and $x > 0$ when $x_i > 0$ for all i, $1 \leqslant i \leqslant n$. We will say that a matrix is nonnegative (nonpositive) in the event that all of its entries are nonnegative (nonpositive). A symmetric matrix is positive semidefinite if $x^{\mathrm{T}}Ax \geqslant 0$, for all $x \in \mathbf{R}^n$, and positive definite if $x^{\mathrm{T}}Ax > 0$, for all $x \in \mathbf{R}^n$, $x \neq 0$. A symmetric matrix $A \in \mathbf{R}^{n\times n}$ is said to be copositive when $x^{\mathrm{T}}Ax \geqslant 0$ for all $x \geqslant 0$, and A is said to be strictly copositive when $x^{\mathrm{T}}Ax > 0$ for all $x \geqslant 0$ and $x \neq 0$. A nonnegative matrix is a copositive matrix, as is a positive semidefinite matrix. Clearly, the sum of two copositive matrices is copositive. It was shown by Alfred Horn (see [2]) that a copositive matrix need not be the sum of a positive semidefinite matrix and a nonnegative matrix.

2. Conditions for copositivity of block matrices

Lemma 1, which is an extension of a similar result for positive semidefinite matrices, appeared in [4,6,9]. We include a proof for completeness.

Lemma 1. *Let $A \in \mathbf{R}^{n\times n}$ be copositive. If $x_0 \geqslant 0$ and $x_0^{\mathrm{T}}Ax_0 = 0$, then $Ax_0 \geqslant 0$.*

Proof. Let $\epsilon > 0$. Then for any i, $1 \leqslant i \leqslant n$, since $x_0 + \epsilon e_i \geqslant 0$, we have

$$(x_0 + \epsilon e_i)^{\mathrm{T}}A(x_0 + \epsilon e_i) = x_0^{\mathrm{T}}Ax_0 + 2\epsilon e_i^{\mathrm{T}}Ax_0 + \epsilon^2 e_i^{\mathrm{T}}Ae_i \geqslant 0. \tag{1}$$

This says that $2\epsilon e_i^{\mathrm{T}}Ax_0 \geqslant -\epsilon^2 a_{ii}$, so $e_i^{\mathrm{T}}Ax_0 \geqslant -\frac{\epsilon}{2}a_{ii}$. But this is true for any $\epsilon > 0$, so $Ax_0 \geqslant 0$. $\square$

A form of Theorem 2, restricted to when $b \geqslant 0$, or $b \leqslant 0$, was given in [1].

Theorem 2. *Let $A = \begin{pmatrix} a & b^{\mathrm{T}} \\ b & A' \end{pmatrix} \in \mathbf{R}^{n\times n}$, where $A' \in \mathbf{R}^{(n-1)\times(n-1)}$, $b \in \mathbf{R}^{n-1}$, and $a \in \mathbf{R}$. Then A is copositive if and only if $a \geqslant 0$; A' is copositive; if $a > 0$ then $x'^{\mathrm{T}}\left(A' - \frac{bb^{\mathrm{T}}}{a}\right)x' \geqslant 0$, for all $x' \in \mathbf{R}^{n-1}$, such that $x' \geqslant 0$ and $b^{\mathrm{T}}x' \leqslant 0$; if $a = 0$ then $b \geqslant 0$.*

Proof. For $x = (x_1, x')^{\mathrm{T}} \in \mathbf{R}^n$, where $x_1 \in \mathbf{R}$ and $x' \in \mathbf{R}^{n-1}$, we have

$$x^{\mathrm{T}}Ax = ax_1^2 + 2b^{\mathrm{T}}x'x_1 + x'^{\mathrm{T}}A'x', \tag{2}$$

$$= a\left[x_1 + \frac{b^{\mathrm{T}}x'}{a}\right]^2 + x'^{\mathrm{T}}\left(A' - \frac{bb^{\mathrm{T}}}{a}\right)x' \quad (\text{if } a > 0). \tag{3}$$

Suppose A is copositive. Then evidently $a \geqslant 0$ and A' is copositive. If $a > 0$ and $b^{\mathrm{T}}x' \leqslant 0$ then taking $x_1 = -\frac{b^{\mathrm{T}}x'}{a}$ we have that $x'^{\mathrm{T}}\left(A' - \frac{bb^{\mathrm{T}}}{a}\right)x' \geqslant 0$. If $a = 0$ then from Lemma 1 with $x_0 = e_1$, we have $b \geqslant 0$.