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LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 394 (2005) 109–118
www.elsevier.com/locate/laa

# Generalized Shannon inequalities based on Tsallis relative operator entropy

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Received 2 June 2004; accepted 30 June 2004

Submitted by R.A. Brualdi

**Abstract**

Tsallis relative operator entropy is defined and then its properties are given. Shannon inequality and its reverse one in Hilbert space operators derived by Furuta [Linear Algebra Appl. 381 (2004) 219] are extended in terms of the parameter of the Tsallis relative operator entropy. Moreover the generalized Tsallis relative operator entropy is introduced and then several operator inequalities are derived.


*AMS classification:* 47A63; 60E15; 26D15

*Keywords:* Operator inequality; Tsallis relative operator entropy; Shannon inequality

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0024-3795/$ - see front matter 
doi:10.1016/j.laa.2004.06.025

## 1. Introduction

Tsallis entropy

$$S_q(X) = -\sum_x p(x)^q \ln_q p(x)$$

was defined in [6] for the probability distribution $p(x)$, where $q$-logarithm function is defined by $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ for any nonnegative real numbers $x$ and $q \neq 1$. It is easily seen that Tsallis entropy is one parameter extension of Shannon entropy $S_1(X) \equiv -\sum_x p(x) \log p(x)$ and converges to it as $q \to 1$. The study based on Tsallis type entropies has been developed in mainly statistical physics [7]. In the recent work [1], Tsallis type relative entropy in quantum system, defined by

$$D_q(\rho|\sigma) \equiv \frac{1}{1-q}[1 - Tr(\rho^q \sigma^{1-q})] \tag{1}$$

for two density operators $\rho$ and $\sigma$ (i.e., positive operators with unit trace) and $0 \leqslant q < 1$, was investigated.

On the other hand, the relative operator entropy was defined by Fujii and Kamei [3]. Many important results in operator theory and information theory have been published in the relation to Golden–Thompson inequality [2,5]. We are interested in not only the properties of the Tsallis type relative entropy but also the properties before taking a trace, namely, Tsallis type relative operator entropy which is a parametric extension of the relative operator entropy. In this paper, we define the Tsallis relative operator entropy and then show some properties of Tsallis relative operator entropy. To this end, we slightly change the parameter $q$ in Eq. (1) to $\lambda$ in our definition which will be appeared in the following section. Moreover, in order to make our definition correspond to the definition of the relative operator entropy defined in [3], we change the sign of the original Tsallis relative entropy.

## 2. Tsallis relative entropy

As mentioned above, we adopt the slightly modified definition of the Tsallis relative entropy in the following.

**Definition 1.** Let $a = \{a_1, a_2, \ldots, a_n\}$ and $b = \{b_1, b_2, \ldots, b_n\}$ be two probability vectors satisfying $a_j, b_j > 0$. Then for $0 < \lambda \leqslant 1$

$$S_\lambda(a|b) = \frac{\sum_{j=1}^n a_j^{1-\lambda} b_j^\lambda - 1}{\lambda} \tag{2}$$

is called Tsallis relative entropy between $a$ and $b$.