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A linear algebra approach to the conjecture of Collatz

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Abstract

We show that a “periodic” version of the so-called conjecture of Collatz can be reformulated in terms of a determinantal identity for certain finite-dimensional matrices M_k, for all $k \geqslant 2$. Some results on this identity are presented. In particular we prove that if this version of the Collatz's conjecture is false then there exists a number k satisfying $k \equiv 8 \pmod{18}$ for which the orbit of $\frac{k}{2}$ is periodic.

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1. Introduction

The Collatz's conjecture is a well-known open problem. This is also quoted in the literature as the $3n + 1$ problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and Ulam's problem. In its traditional formulation the conjecture says that

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any orbit for the iteration function $f(n) = (3n + 1)/2$, for n odd, and $f(n) = n/2$, for n even, is always attracted to the value 1. A comprehensive discussion on the subject, namely equivalent formulations and related problems arising in several branches of mathematics, can be found in [4, Chapter 1] and [1,2] where relevant literature is discussed.

In this paper we only consider a "periodic" version of the Collatz conjecture which can be stated as: The unique periodic orbit of the function f is the orbit of 1. For convenience, hereafter we refer to this conjecture as being the Collatz's conjecture.

We show how to translate the conjecture of Collatz in terms of the determinant of finite-dimensional matrices denoted by M_k. Namely, in Section 2 we show that the conjecture is true if and only if $\det M_k = 1 - x^2$, for all $k \geqslant 2$.

Using elementary Linear Algebra we prove in Section 3 that $\det(M_k) = \det(M_{k-1})$ for all $k \not\equiv 8 \pmod{18}$. A tentative proof of the equality $\det M_k = 1 - x^2$, for all $k \geqslant 2$ is outlined. Unfortunately this proof is not complete since we were not able to prove that a certain tree rooted at $\frac{2k-1}{3}$, with $k \equiv 8 \pmod{18}$, has not a vertex equal to $\frac{k}{2}$. This tree is constructed in order to produce a nontrivial solution for a certain homogeneous system $\tilde{M}_{k-1}Y = 0$, when $k \equiv 8 \pmod{18}$. Let us emphasize that if such a solution does exist, then $\det(\tilde{M}_{k-1}) = 0$ and the Collatz's conjecture is true. Nevertheless, we can prove that such nontrivial solution exists for certain values of k and we give an algorithm for its construction (see Section 4). As a consequence of the discussion on the existence of this solution we show that if the Collatz's conjecture is false then there exist integers $k \equiv 8 \pmod{18}$ for which the orbit of $\frac{k}{2}$ is periodic.

2. Periodic orbits and a Jacobi formula

Consider the following Collatz's iteration function defined on the set of positive integers $\mathbb{N}$ (see [4, p. 11]):

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

For $n \in \mathbb{N}$, the orbit of n is the sequence defined as $\mathcal{O}_n = \{f^j(n) : j \geqslant 0\}$, where $f^j = f \circ f^{j-1}$ is the j-fold iterate of f. An orbit $\mathcal{O}_n$ is said to be periodic, with period $p \geqslant 1$, if p is the least positive integer verifying $f^p(n) = n$. For instance, for f given by (1) the orbit of 1 is periodic with period 2 (since $f(1) = 2$ and $f^2(1) = 1$).

A "periodic" version of the so-called Collatz's conjecture can be stated as: "$\mathcal{O}_1$ is the unique periodic orbit of f". Note that $\mathcal{O}_1 = \mathcal{O}_2 = \{1, 2\}$. Throughout this paper each time we refer to the conjecture of Collatz we mean this version of the conjecture.

For each $m \geqslant 2$, consider the following $m \times m$ matrix A_m, whose entries $A_m(i, j)$ are defined by