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On graphs with prescribed star complements[☆]

Xiying Yuan, Qingqing Zhao, Lele Liu[*], Hongyan Chen

Department of Mathematics, Shanghai University, Shanghai 200444, PR China

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Let μ be an eigenvalue of a simple graph G with multiplicity $k \geq 1$. A star complement for μ in G is an induced subgraph of G of order $n-k$ with no eigenvalue μ. In this paper, we study the maximal graphs with the star S_m as a star complement for -2. The maximal graphs with S_3, S_4, S_{13} and S_{21} as a star complement for -2 are described. We also describe the regular graphs with $K_{2,s}$ $(s \geq 2)$ as a star complement for an eigenvalue μ.

1. Introduction

Let G be a simple graph with vertex set $V(G) = [n] := \{1, 2, \dots, n\}$. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if vertex i is adjacent to vertex j, and 0 otherwise. We use the notation $i \sim j$ to indicate that i, j are adjacent. The eigenvalues of G are just the eigenvalues of $A(G)$, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\mathcal{E}(\mu)$ be the eigenspace of $A(G)$ for an eigenvalue μ. (For more details on graph spectra, see [1].)

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* Corresponding author.
E-mail addresses: xiyingyuan2007@hotmail.com (X. Yuan), ahhylau@gmail.com (L. Liu).

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Let μ be an eigenvalue of G with multiplicity k. A *star set* for μ in G is a subset X of $V(G)$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$, where $G - X$ is the subgraph of G induced by $\overline{X} := V(G) \setminus X$. In this situation $H := G - X$ is called a *star complement* corresponding to μ. Star sets and star complements exist for any eigenvalue of a graph, and they need not to be unique. The basic properties of star sets are established in Chapter 7 of [2].

In [5], it was proved that if $Y \subset X$ then $X \setminus Y$ is a star set for μ in $G - Y$; thus the induced subgraph $G - Y$ also has H as a star complement for μ. If G has H as a star complement for μ, and G is not a proper induced subgraph of some other graph with H as a star complement for μ, then G is a *maximal graph* with star complement H for μ (we also say that it is an H*-maximal graph* for μ). In general, there will be various different maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph.

In 2007, Stanić [11] investigated the maximal graphs with star S_m as a star complement for $\lambda_2 = 1$. In Section 3 of this paper, we study the maximal graphs with S_m as a star complement for -2. We prove that S_3, S_4, S_{13}, S_{14}, S_{21}, S_{23} and S_{30} are the only stars which can be star complements for -2. Also the maximal graphs with S_3, S_4, S_{13} and S_{21} as star complements for -2 are described. For $H = S_3$, there is a unique maximal graph. For $H = S_4$, there are two non-isomorphic maximal graphs. For $H = S_{13}$ and S_{21}, there are various different maximal graphs, possibly of different orders.

In 1999, Jackson and Rowlinson [8] studied all regular graphs with $K_{2,5}$ as a star complement. In 2010, Rowlinson and Tayfeh-Rezaie [9] discussed the regular graphs with star $K_{1,s}$ as a star complement. More generally, Rowlinson [10] considered the bipartite graphs with complete bipartite $K_{r,s}$ as a star complement in 2014. In Section 4, we will study the regular graphs with $K_{2,s}$ ($s \geq 2$) as a star complement for an eigenvalue μ.

2. Preliminaries

In this section we note properties of star sets and star complements that will be required in the sequel. The following result, known as the Reconstruction Theorem (and its converse), is fundamental to the theory of star complements.

Theorem 2.1 *([2]). Let X be a set of k vertices in the graph G. Suppose that G has adjacency matrix*

$$\begin{pmatrix} A_X & B^{\mathrm{T}} \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and

$$\mu I - A_X = B^{\mathrm{T}}(\mu I - C)^{-1}B. \tag{2.1}$$