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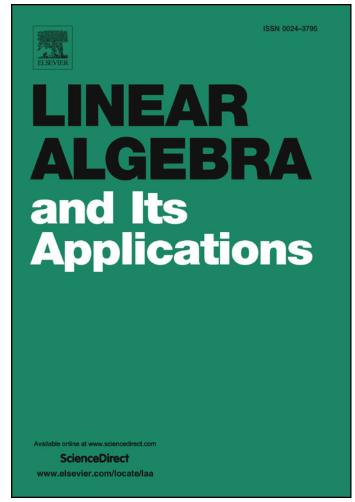

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GRADINGS ON SEMISIMPLE ALGEBRAS

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Abstract. The classification of gradings by abelian groups on finite direct sums of simple finite-dimensional nonassociative algebras over an algebraically closed field is reduced, by means of the use of loop algebras, to the corresponding problem for simple algebras. This requires a good definition of (free) products of group-gradings.

1. Introduction

In 1989, Patera and Zassenhaus [PZ89] undertook a systematic study of gradings by abelian groups on finite-dimensional simple Lie algebras over the complex numbers, with fine gradings as the central objects. A key example of fine grading is the root space decomposition of a finite-dimensional semisimple Lie algebra relative to a Cartan subalgebra, but there are many other fine gradings that reflect the symmetries of these algebras. A description of fine gradings on the classical simple Lie algebras (other than D_4, which is exceptional in many aspects) over $\mathbb{C}$ followed in [HPP98]. The classification of fine gradings on all finite-dimensional simple Lie algebras over an algebraically closed field has been recently completed through the efforts of many authors: see [EK13, Yu16, Eld16]. A survey of the main ideas and results appears in [DE16].

Many of the gradings that appear are related to gradings on simple associative or Jordan algebras, and the gradings by abelian groups on finite-dimensional simple Lie, associative, or Jordan algebras have also been classified in the last years (see [EK13] and the references therein). If the grading group becomes part of the definition of the grading, then the right classification is the classification of G-gradings up to isomorphism. On the other hand, any grading on a finite-dimensional algebra is a coarsening of a fine grading, so fine gradings, like the root space decomposition mentioned above, become the key objects of study and of classification up to equivalence.

Time is ripe to try to extend the known classifications on simple algebras to semisimple algebras. Usually, the word 'semisimple' refers to a given 'radical' being zero. However, in many cases of interest, like finite-dimensional Lie algebras over fields of characteristic zero, or finite-dimensional associative or Jordan algebras, the semisimple algebras are just the finite direct sums of simple algebras. In this paper, we will take this restricted definition of 'semisimple algebras'.

Therefore, in what follows, a *semisimple algebra* will refer to a finite direct sum of simple algebras. The word 'algebra' will refer to a vector space over a ground field $\mathbb{F}$, endowed with a bilinear multiplication. No other restriction will be imposed, so

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