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The complete characterization of the minimum size supertail in a subspace partition

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ABSTRACT

Let q be a prime power and let n be a positive integer. Let $V = V(n, q)$ denote the vector space of dimension n over $\mathbb{F}_q$. A *subspace partition* $\mathcal{P}$ of V is a collection of subspaces of V with the property that each nonzero vector is in exactly one of the subspaces in $\mathcal{P}$. Suppose that $d_1, \ldots, d_k$ are the different dimensions, in increasing order, that occur in the subspace partition $\mathcal{P}$. For any integer s, with $2 \le s \le k$, the d_s*-supertail* $\mathcal{S}$ of $\mathcal{P}$ is the collection of all subspaces $X \in \mathcal{P}$ such that $\dim X < d_s$. It was shown that $|\mathcal{S}| \ge \sigma_q(d_s, d_{s-1})$, where $\sigma_q(d_s, d_{s-1})$ denotes the minimum number of subspaces over all subspace partitions of $V(d_s, q)$ in which the largest subspace has dimension d_{s-1}. Moreover, it was shown that if $d_s \ge 2d_{s-1}$ and equality holds in the previous bound on $|\mathcal{S}|$, then the union of the subspaces in $\mathcal{S}$ forms a subspace. This characterization was also conjectured to hold if $d_s < 2d_{s-1}$. This conjecture was recently proved in certain cases. In this paper, we use a much simpler approach to completely settle this conjecture.

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1. Introduction

Let q be a prime power and let n be a positive integer. Let $V = V(n,q)$ denote the vector space of dimension n over $\mathbb{F}_q$. A subspace of dimension t is referred to as a *t-subspace*. A *subspace partition*, or *vector space partition*, $\mathcal{P}$ of V, is a collection of subspaces of V with the property that each nonzero vector is in exactly one of the subspaces in $\mathcal{P}$. A well-known example of a subspace partition is a *spread*, which is a subspace partition in which all subspaces have the same dimension. Pioneering work on spreads has been done by several researchers, e.g., André [1] and Segre [16]. Research work on subspace partitions has also been carried since the early 1900s, e.g., see Heden [7] for a survey. A special feature of subspace partitions is that they naturally occur in various fields such as finite geometry, coding theory, and design theory, e.g., see [1,2,11,12,16] and the references therein.

One main line of research in the area of subspace partitions is the *Classification Problem*, which we shall define after introducing some notation. Given a subspace partition $\mathcal{P}$, of V, the *type* of $\mathcal{P}$ is the multiset that consists of $\dim X$ for all subspaces $X \in \mathcal{P}$. The Classification Problem consists of finding necessary and/or sufficient conditions for a given multiset of integers that is realizable as the type of a subspace partition of V. Although there are many results related to the Classification Problem, e.g., see [3–5,8,13], the main question is still wide open.

Before we describe the main contribution of this paper (Theorem 5), we introduce two necessary conditions and a few more definitions. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ that contains m_{d_i} subspaces of dimension d_i for $1 \le i \le k$. In other words, the type of $\mathcal{P}$ is the multiset that consists of m_{d_i} copies of d_i for $1 \le i \le k$. We denote such a multiset by $d_1^{m_{d_1}} \dots d_k^{m_{d_k}}$. The following necessary conditions are trivial to derive:

$$\sum_{i=1}^{k} m_{d_i}(q^{d_i}-1) = q^n - 1 \qquad (\textit{packing condition}) \tag{1}$$

$$\begin{cases} n \ge d_i + d_j & \text{if } m_{d_i}+m_{d_j} \ge 2 \text{ and } i \ne j; \\ n \ge 2d_i & \text{if } m_{d_i} \ge 2. \end{cases} \qquad (\textit{dimension condition}) \tag{2}$$

Let k, d_i, and m_{d_i} be as defined, and let s be an integer such that $2 \le s \le k$. We define the *d_s-supertail* of $\mathcal{P}$ to be the set of all subspaces $X \in \mathcal{P}$ such that $\dim X < d_s$. For any integers d and t such that $1 \le t \le d$, we also define $\sigma_q(d,t)$ to be the minimum number of subspaces over all subspace partitions of $V(d,q)$ in which the largest subspace has dimension t. It is easy to see that if $t \mid d$, then $\sigma_q(d,t) = (q^d-1)/(q^t-1)$, which is the number of subspaces in a t-spread of $V(d,q)$, i.e., a spread whose subspaces have dimension t. In fact, the exact value of $\sigma_q(d,t)$ is given by the following theorem (see André [1] and Beutelspacher [2] for $d \pmod t \equiv 0$, and see [9,14] for $d \pmod t \not\equiv 0$).

Theorem 1. *Let d, k, t, and r be integers such that $0 \le r < t$, $k \ge 1$, and $d = kt + r$. Then*