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Semilinear equations with exponential nonlinearity and measure data

Équations semi linéaires avec non linéarité exponentielle et données mesures

Daniele Bartolucci [a], Fabiana Leoni [b], Luigi Orsina [b], Augusto C. Ponce [c,d,*]

[a] *Dipartimento di Matematica, Università di Roma "Tre", Largo S. Leonardo Murialdo 1, 00146 Roma, Italy*
[b] *Dipartimento di Matematica, Università di Roma "La Sapienza", Piazza A. Moro 2, 00185 Roma, Italy*
[c] *Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, boîte courier 187, 75252 Paris cedex 05, France*
[d] *Rutgers University, Department of Mathematics, Hill Center, Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA*

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Abstract

We study the existence and non-existence of solutions of the problem

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, and μ is a Radon measure. We prove that if $\mu \leqslant 4\pi\mathcal{H}^{N-2}$, then (0.1) has a unique solution. We also show that the constant 4π in this condition cannot be improved.

Résumé

Nous étudions l'existence et la non existence des solutions de l'équation

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{0.2}$$

* Corresponding author.
E-mail addresses: bartolu@mat.uniroma3.it, bartoluc@mat.uniroma1.it (D. Bartolucci), leoni@mat.uniroma1.it (F. Leoni), orsina@mat.uniroma1.it (L. Orsina), ponce@ann.jussieu.fr, augponce@math.rutgers.edu (A.C. Ponce).

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où Ω est un domaine borné dans $\mathbb{R}^N$, $N \geqslant 3$, et μ est une mesure de Radon. Nous démontrons que si μ vérifie $\mu \leqslant 4\pi\mathcal{H}^{N-2}$, alors le problème (0.2) admet une unique solution. Nous montrons que la constante 4π dans cette condition ne peut pas être améliorée.

MSC: 35J60; 35B05

Keywords: Nonlinear elliptic equations; Exponential nonlinearity; Measure data

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with smooth boundary. We consider the problem

$$\begin{cases} -\Delta u + \mathrm{e}^u - 1 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}(\Omega)$, the space of bounded Radon measures in Ω. We say that a function u is a solution of (1.1) if $u \in L^1(\Omega)$, $\mathrm{e}^u \in L^1(\Omega)$ and the following holds:

$$-\int_\Omega u\Delta\zeta + \int_\Omega (\mathrm{e}^u - 1)\zeta = \int_\Omega \zeta\,\mathrm{d}\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{1.2}$$

Here $C_0^2(\overline{\Omega})$ denotes the set of functions $\zeta \in C^2(\overline{\Omega})$ such that $\zeta = 0$ on $\partial\Omega$. A measure μ is a *good measure* for problem (1.1) if (1.1) has a solution. We shall denote by $\mathcal{G}$ the set of good measures. Problem (1.1) has been recently studied by Brezis, Marcus and Ponce in [1], where the general case of a continuous nondecreasing nonlinearity $g(u)$, with $g(0) = 0$, is dealt with. Applying Theorem 1 of [1] to $g(u) = \mathrm{e}^u - 1$, it follows that for every $\mu \in \mathcal{M}(\Omega)$ there exists a largest good measure $\leqslant \mu$ for (1.1), which we shall denote by μ^*.

In the case $N = 2$, the set of good measures for problem (1.1) has been characterized by Vázquez in [9]. More precisely, a measure μ is a good measure if and only if $\mu(\{x\}) \leqslant 4\pi$ for every x in Ω. Note that any $\mu \in \mathcal{M}(\Omega)$ can be decomposed as

$$\mu = \mu_0 + \sum_{i=1}^{\infty} \alpha_i \delta_{x_i},$$

with $\mu_0(\{x\}) = 0$ for every x in Ω, and δ_{x_i} is the Dirac mass concentrated at x_i. Using Vázquez's result, it is not difficult to check that (see [1, Example 5])

$$\mu^* = \mu_0 + \sum_{i=1}^{\infty} \min\{4\pi, \alpha_i\}\,\delta_{x_i}.$$

This paper is devoted to the study of problem (1.1) in the case $N \geqslant 3$. First of all, let us recall that if μ is a good measure, then (1.1) has a unique solution u (see [1, Corollary B.1]). This solution can be either obtained as the limit of the sequence (u_n) of solutions of

$$\begin{cases} -\Delta u_n + \min\{\mathrm{e}^{u_n} - 1, n\} = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

or as the limit of a sequence (v_n) of solutions of

$$\begin{cases} -\Delta v_n + \mathrm{e}^{v_n} - 1 = \mu_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$