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Global solutions to vortex density equations arising from sup-conductivity

Existence de solutions globales pour des équations de la super-conductivité

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Abstract

In the first part of this paper, we establish the existence of a global renormalized solution to a family of vortex density equations arising from superconductivity. And we show by an explicit example the necessity of the notion of renormalized solution to be used here. In the second part, we prove the global existence and uniqueness of $W^{1,p}$ and C^α solutions to a modified model, which is derived from the physically sign-changing vortices case.

Résumé

On montre l'existence de solutions globales pour une famille d'équations provenant de la super-conductivité. On montre par un exemple que la notion de solutions renormalizées est nécessaire ici. Dans la seconde partie de ce papier, on montre l'existence et l'unicité de solutions $W^{1,p}$ et C^α pour un modèle qui décrit des vortex qui changent de signes.

MSC: 35Q35; 35M10

Keywords: Superconductivity; Vortex density; Hydrodynamics; Young measure; Renormalized solutions

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1. Introduction

This paper deals with two models coming from the hydrodynamic equations of Ginzburg–Landau vortices (see [13,6] for some earlier related works). In the first part of this paper, we shall establish the global existence of renormalized solutions to

$$\begin{cases} \partial_t\rho + \mathrm{div}(u\rho) = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}^2, \\ u = \nabla\Delta^{-1}\rho, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{1.1}$$

with initial data $\rho_0 \in L^1(\mathbb{R}^2)$.

Our main motivation to study this problem comes from the type-II superconductivity. It is generally accepted that, when effects due to thermal or field fluctuations are taken into account, the Abrikosov vortex lattice obtained from the mean-field theory can melt and form a vortex liquid. Then one of the important issues that one wishes to understand is the intrinsic nonlinear effects in the dynamics of such a liquid, where the vortex density satisfies (1.1). The rigorous finite gradient vortex dynamics was studied in [12] (see also [10]). The formal derivation of (1.1) from the finite vortex dynamics was carried out in [19] (see also [1]). Under the assumption that ρ_0 is a positive Randon measure, the authors in [13] mathematically justified the formal derivation. One can check more physical explanation to (1.1) from [19,1,13].

When we take a complex time relaxation in the finite gradient dynamics into account, we need to rotate the second equation of (1.1), and then the equation is modified to the following form:

$$\begin{cases} \partial_t\rho + \mathrm{div}(u\rho) = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}^2, \\ u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \nabla\Delta^{-1}\rho, \\ \rho|_{t=0} = \rho_0. \end{cases} \tag{1.2}$$

Indeed, when $\cos\theta = 0$, (1.2) is the classical 2-D vorticity-formulated incompressible Euler equation. In that case, with smooth initial data, (1.2) has a unique global smooth solution. When $\rho_0 \in L^\infty$, Yudovich [21] solves the global existence and uniqueness of weak solutions to (1.2). In [5] and [18], the authors establish the global existence of weak solutions to (1.2) with $\rho_0 \in L^p$ for $1 \leqslant p < \infty$. However, the uniqueness of the weak solutions in this class is still open. When $\rho_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$, the above problem is the so-called vortex sheets problem in fluid mechanics. In 1991, Delort [7] solved the existence problem when ρ_0 keeps the sign, the remaining case is still an outstanding open question in the mathematical fluid mechanics.

Compared to the 2-D incompressible Euler equations, when $\cos\theta \neq 0$, smooth solution to (1.2) may blow up in finite time. In [13], the authors proved a global existence result to (1.1) when $\rho_0 \in \mathcal{M}^+(\mathbb{R}^2)$. Furthermore, they found that $\rho(t,x)$ will be a function for $t > 0$ and belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ with $p < 2$. However, when ρ_0 changes sign, the second author and his collaborator [6] found that there exists concentration phenomena in the approximate solutions sequence of (1.2) no matter how smooth the initial data is. This argument implies the global existence of a measure-valued solution to (1.2). This motivates us to think that: to make the measure-valued solution more precise, we may need the notion of renormalized solution, which was first introduced by the DiPerna and Lions [3,4] in the study of transport and kinetic equations, for (1.2). On the other hand, it is easy to observe that (1.2) will keep the L^1 norm of ρ nondecreasing with respect to t even after formation of singularities to the smooth solution. Hence it is natural to study the global existence for the system (1.2) with initial data in L^1. As we allow ρ_0 to change sign, the proof of the global existence of the renormalized solution to (1.2) makes no difference for $\cos\theta \neq 0$. For simplicity, we take $\cos\theta = 1$, which reduces to (1.1).

Note that the notion of renormalized solution can allow concentration in the solution, but it will make the problem much more nonlinear than the original problem. Then the main issue in the proof of the existence is to prove that there is no oscillation in the approximate solution sequence. Motivated by [14,15,22] and [11], we will use L^p Young measure theory (see [20,17] and [16]) to cancel the possible oscillations in the approximate solutions.